MATH 211 HOMEWORK ASSIGNMENT 7 SOLUTIONS

FALL 2006, WILLIAMS COLLEGE

Abstract. These are the instructor's solutions.

1. Nonsingularity

Show that the composition of two nonsingular linear transformations is also nonsingular.

1.1. **Solution.** A linear transformation is nonsingular when its nullity is zero, that is, when its kernel is the trivial subspace $\{0\}$. (Note that this is equivalent to injectivity!) Let $T : U \to V$ and $S : \to W$ be the linear transformations in question. Suppose that $u \in U$ lies in the kernel of $S \circ T$. We must show that $u = 0$. Since $u \in \ker(S \circ T)$ we see that, as elements of W,

$$0 = (S \circ T)(u) = S(T(u)).$$

Thus, $T(u)$ lies in the kernel of S. Since S is nonsingular, we deduce that $T(u) = 0 \in V$. This means that u lies in the kernel of T. Since T is nonsingular, we see that $u = 0$.

Therefore, the null space of $S \circ T$ consists of only the vector $u = 0$, and hence $S \circ T$ is nonsingular.

2. Rank-Nullity

Find bases of the kernel and image and verify the Rank-Nullity theorem for the linear transformation $T_A : \mathbb{R}^5 \to \mathbb{R}^4$ associated to the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 & 2 & -2 \\ 1 & 0 & 3 & 5 & 1 \\ 6 & 2 & 1 & 0 & -9 \\ -3 & 3 & -1 & 7 & 8 \end{pmatrix}.$$

2.1. **Solution.** We put A into reduced row echelon form:

$$A \sim \begin{pmatrix} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So the kernel is

$$\ker(T_A) = \text{null}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\},$$

and the image is

$$\operatorname{im}(T_A) = \operatorname{col}(A) = \operatorname{span}\left\{ \begin{pmatrix} 2 \\ 1 \\ 6 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

We see by construction that these sets are bases for the relevant spaces, so

$$\operatorname{rank}(T_A) + \operatorname{nullity}(T_A) = 3 + 2 = 5 = \dim(\mathrm{R}^5)$$

which verifies the result of the rank-nullity theorem in this case.

3. Hyperplanes

Is it possible to find a family of 4 hyperplanes in $\mathbb{R}^4$ so that any subset of three hyperplanes must intersect in exactly a line, but that the common intersection of all four hyperplanes is empty? If so, give a concrete example. If not, explain why it is not possible in terms of some of the tools we have developed.

3.1. **Solution.** This is possible, but it can be tricky to see how at first. Suppose that the hyperplanes are defined by normal vectors $n_1, n_2, n_3, n_4 \in \mathbb{R}^4$. (We'll view these as column vectors.) That is, the ith hyperplane is the set of solutions to an equation $\langle n_i, x \rangle = b_i$.

Take for example the first three hyperplanes. Their common intersection is the solution set to a matrix equation $Ax = b$ where $A = \begin{pmatrix} n_1^t \\ n_2^t \\ n_3^t \end{pmatrix}$. To have exactly a line's worth of solutions, we must have $\operatorname{nullity}(A) = 1$. So by the rank-nullity theorem, $\operatorname{rank}(A) = 4 - \operatorname{nullity}(A) = 3$. Since the matrix A has three rows, we see that these rows are linearly independent.

This is true for any choice of three hyperplanes. But to get no solution for the intersection of four planes, we have to have

$$\operatorname{rank}\begin{pmatrix} n_1^t \\ n_2^t \\ n_3^t \\ n_4^t \end{pmatrix} < \operatorname{rank}\begin{pmatrix} n_1^t & b_1 \\ n_2^t & b_2 \\ n_3^t & b_3 \\ n_4^t & b_4 \end{pmatrix}.$$

Now the matrix on the right has to have rank at least 3 (since it contains our A above as a submatrix). And if it were 4, then it would be impossible to have the inequality, since each matrix has 4 rows. So, we must arrange things so that the four vectors are linearly dependent, but any subset of three of them is linearly independent.

I'll use $n_1 = e_1, n_2 = e_2, n_3 = e_3, n_4 = e_1 + e_2 + e_3$. All that is left is to choose the b_i's. This is not so tricky now that we understand what we're up against. I will use

$$\begin{aligned} \mathcal{H}_1 &= \{x \in \mathbb{R}^4 \mid x_1 = 1\} \\ \mathcal{H}_2 &= \{x \in \mathbb{R}^4 \mid x_2 = 1\} \\ \mathcal{H}_3 &= \{x \in \mathbb{R}^4 \mid x_3 = 1\} \\ \mathcal{H}_4 &= \{x \in \mathbb{R}^4 \mid x_1 + x_2 + x_3 = 6\}. \end{aligned}$$

It is not difficult to verify that this set of planes has the required properties.

4. An interesting new space

Let $\hom(\mathbb{R}^3, \mathbb{R}^4)$ be the collection of all linear mappings from $\mathbb{R}^3$ into $\mathbb{R}^4$.

- What should addition and scalar multiplication be in this space?
- Show that $\hom(\mathbb{R}^3, \mathbb{R}^4)$ is a vector space under these operations.
- Find a basis for $\hom(\mathbb{R}^3, \mathbb{R}^4)$ and use it to compute the dimension of this space.
- Describe two different, more easily understood spaces which are isomorphic to $\hom(\mathbb{R}^3, \mathbb{R}^4)$. Write down explicit isomorphisms between $\hom(\mathbb{R}^3, \mathbb{R}^4)$ and the two spaces.

4.1. **Solution.** Let S and T be elements of $\hom(\mathbb{R}^3, \mathbb{R}^4)$, and α a real number (that is, a scalar). We define the sum $S+T$ to be the function

$$(S+T)(x) = S(x) + T(x),$$

where we take the sum on the right in $\mathbb{R}^4$, and we define the scalar multiple αT to be the function

$$(\alpha T)(x) = \alpha T(x),$$

where the scalar multiplication on the right is happening in $\mathbb{R}^4$.

We must also check that $S+T$ and αS are elements of $\hom(\mathbb{R}^3, \mathbb{R}^4)$, that is that they are linear transformations. For any vectors $v, w \in \mathbb{R}^3$ and scalars $a, b \in \mathbb{R}$ we have

$$\begin{aligned}(S+T)(av+bw) &= S(av+bw) + T(av+bw) = aS(v) + bS(w) + aT(v) + bT(w) \\ &= a(S(v)+T(v)) + b(S(w)+T(w)) = a(S+T)(v) + b(S+T)(w) \\ \text{and} \\ (\alpha S)(av+bw) &= \alpha(S(av+bw)) = \alpha(aS(v)+bS(w)) \\ &= a(\alpha S(v)) + b(\alpha S(w)) = a(\alpha S)(v) + b(\alpha S)(w),\end{aligned}$$

hence both these maps are linear transformations.

The fact that this is a vector space follows just like in the problem from the last assignment. You have a set of functions which have a target equal to a vector space, so all of the vector space rules get inherited by the set of functions. It is a bit tedious to type one more time, but you should write this part out for practice. Recall that there are eight rules to check. Just for clarity in what we do below, note that the zero element, Z, of $\hom(\mathbb{R}^3, \mathbb{R}^4)$ is the function defined by

$$Z(x) = 0 \text{ for all } x \in \mathbb{R}^3.$$

To find a basis for our vector space, we first try to find a spanning set. Recall that we have canonical bases $\{e_1, e_2, e_3\}$ for $\mathbb{R}^3$ and $\{e_1', e_2', e_3', e_4'\}$ for $\mathbb{R}^4$. I'm using primes for the second set to avoid confusion with the first set, there is nothing new going on.

We know that every linear transformation is determined by what it does to a basis of the domain. So each linear transformation is determined by what it does to the set $\{e_1, e_2, e_3\}$. That is, a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ is defined by the three vectors $T(e_1)$, $T(e_2)$, $T(e_3)$. Alternately, we can think of it as spanned by the 12 constants which appear in the vectors when expanded like below:

$$\begin{aligned} T(e_1) &= a_{11}e'_1 + a_{21}e'_2 + a_{31}e'_3 + a_{41}e'_4 \\ T(e_2) &= a_{12}e'_1 + a_{22}e'_2 + a_{32}e'_3 + a_{42}e'_4 \\ T(e_3) &= a_{13}e'_1 + a_{23}e'_2 + a_{33}e'_3 + a_{43}e'_4. \end{aligned} \tag{1}$$

Thus, if we let T_{ij} be the linear mapping which carries e_j to e'_i and the other e_k's to zero, we see that

$$\hom(\mathbb{R}^3, \mathbb{R}^4) = \operatorname{span}\{T_{ij} \mid i = 1, 2, 3, j = 1, 2, 3, 4\}.$$

Now suppose that we have a linear combination of our spanning set which represents the zero vector Z:

$$\sum_{i=1,2,3,4} \sum_{j=1,2,3} a_{ij} T_{ij} = Z. \tag{2}$$

If we evaluate this equation at the vector e_1 we see that

$$\begin{aligned} 0 =& Z(e_1) = \sum_{i=1,2,3,4} \sum_{j=1,2,3} a_{ij} T_{ij}(e_1) \\ &= a_{11}e'_1 + a_{21}e'_2 + a_{31}e'_3 + a_{41}e'_4 + 0 \end{aligned}$$

Since the set $\{e'_1, e'_2, e'_3, e'_4\}$ is linearly independent, we deduce that $a_{11} = a_{12} = a_{13} = a_{14} = 0$. Similarly, by evaluating equation (2) at e_2 and e_3, we see that all of the other coefficients are also zero. Thus, $\hom(\mathbb{R}^3, \mathbb{R}^4)$ has the set of T_{ij}'s as a basis, and therefore has dimension 12.

Since $\hom(\mathbb{R}^3, \mathbb{R}^4)$ a 12 dimensional vector space over the real numbers, we know that it is isomorphic to $\mathbb{R}^{12}$ using the coordinate representation map with respect to the basis $\{T_{11}, T_{12}, \dots, T_{42}, T_{43}\}$. Explicitly, this is the map which sends a linear transformation T to the vector $\begin{pmatrix} a_{11} & a_{12} & \dots & a_{42} & a_{43} \end{pmatrix}^t$, where the coefficients are chosen like in equations (1), above.

The other representation is also hinted at by the equations (1). The space $\hom(\mathbb{R}^3, \mathbb{R}^4)$ is isomorphic to the vector space

$$M_{43}(\mathbb{R}) = \{\text{real } 4 \times 3 \text{ matrices}\}.$$

The isomorphism is given by the linear mapping

$$M : T \mapsto M(T) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}.$$

That this defines a linear map which is injective and surjective is a simple direct check using the theorem about describing a linear transformation by its action on a basis. (Only this time M is the transformation, and T_{ij} is the basis.)