MATH 211 HOMEWORK ASSIGNMENT 8

FALL 2006, WILLIAMS COLLEGE

Abstract. This assignment has 6 problems on 2 pages. It is due Wednesday, November 21 by 5pm. Don't hesitate to ask for help. This may seem long, but it is important to do this stuff now, and not after a break.

1. A Property of the transpose

Let A and B be square matrices. Show that $(AB)^t = B^tA^t$.

1.1. Solution.

Let A be the matrix with entries $\big(a_{ij}\big)$, and B the matrix with entries $\big(b_{ij}\big)$. Then AB has entries $c_{ij} = \sum_k a_{ik}b_{kj}$.

Similarly, we see that A^t is the matrix with entries $\big(d_{ij}\big)$, where $d_{ij} = a_{ji}$, and B^t is the matrix with entries $\big(e_{ij}\big)$, where $e_{ij} = b_{ji}$. So now B^tA^t has entries

$$f_{ij} = \sum_k e_{ik}d_{kj} = \sum_k b_{ki}a_{jk} = \sum_k a_{jk}b_{ki} = c_{ji}.$$

Thus $B^tA^t = C^t = (AB)^t$.

2. Some more about Bases

Let $T : V \to W$ be an invertible linear transformation, and let $B = \{v_1, \dots, v_n\}$ be a basis of V. Show that the set $T(B) = \{T(v_1), \dots, T(v_n)\}$ is a basis of W.

2.1. Solution.

The image of T is spanned by the set $T(B)$, but since T is an isomorphism, $\operatorname{im}(T) = W$. Thus $W = \operatorname{span}(T(B))$. So all that remains is to show that $T(B)$ is a linearly independent set. Consider a general linear combination of the elements which represent the zero vector:

$$0 = a_1T(v_1) + \cdots + a_nT(v_n) = T(a_1v_1 + \cdots + a_nv_n).$$

Since T is an isomorphism, $\ker T = \{0\}$ and hence

$$a_1v_1 + \cdots + a_nv_n = 0.$$

Since $\{v_1, \dots v_n\}$ is a basis, we conclude that $a_1 = \cdots = a_n = 0$. Therefore, $T(B)$ is linearly independent.

3. Matrix representations

Let $T = T_A : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation determined by the matrix

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 7 & 4 \\ 1 & 4 & 3 \end{pmatrix}.$$

Find the matrix representations of T with respect to each of the following bases

(1) $B_1 = \{u_1 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^t, u_2 = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}^t, u_3 = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}^t\}$,
(2) $B_2 = \{v_1 = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^t, v_2 = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^t, v_3 = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}^t\}$,
(3) $B_3 = \{w_1 = \begin{pmatrix} -3 & 2 & 1 \end{pmatrix}^t, w_2 = \begin{pmatrix} 1 & 1 & 3 \end{pmatrix}^t, w_3 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^t\}$.

3.1. **Solution.** For this and the next three problems, there is the standard way to compute the answers, and a nice trick. Since I only outlined the trick briefly at the end of class, I will write the solutions this way now as further explanation. One can still do the problem totally from scratch by computing with the definitions.

It is important to notice that this setup only works because we are working with a matrix map in $\mathbb{R}^n$.

Let $B_0 = \{e_1, e_2, e_3\}$ be the canonical basis of $\mathbb{R}^3$. Then by the change of basis theorem we have

$$[T_A]_{B_i} = P_i^{-1}[T_A]_{B_0}P_i$$

where P_i is the change of basis matrix from B_0 to B_i. Since B_0 is the standard basis, we know two important facts, both of which can be checked easily:

- $[T_A]_{B_0} = A$
- P_i is the matrix with columns given by the ordered basis vectors in B_i.

With these in mind, we compute that

$$P_1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad P_3 = \begin{pmatrix} -3 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{pmatrix}.$$

Thus, we see that

$$\begin{aligned} [T_A]_{B_1} &= P_1^{-1}[T_A]_{B_0}P_1 \\ &= \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 3 & 1 \\ 2 & 7 & 4 \\ 1 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 10 & 8 & 20 \\ 13 & 11 & 28 \\ -5 & -4 & -10 \end{pmatrix}, \\ [T_A]_{B_2} &= P_1^{-1}[T_A]_{B_0}P_1 \\ &= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 3 & 1 \\ 2 & 7 & 4 \\ 1 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 2 & 4 \\ 0 & 0 & 0 \\ 5 & 4 & 7 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned}[T_A]_{B_3} &= P_1^{-1}[T_A]_{B_0}P_1 \\ &= \begin{pmatrix} -3 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 3 & 1 \\ 2 & 7 & 4 \\ 1 & 4 & 3 \end{pmatrix} \begin{pmatrix} -3 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 8/5 & 14/5 & 3/5 \\ 44/5 & 77/5 & 14/5 \\ -20 & -35 & -6 \end{pmatrix}.\end{aligned}$$

4. Changing Basis

Let B_1, B_2, B_3 be as in the last problem. Find the change of basis matrices from B_1 to B_2, from B_2 to B_3 and from B_1 to B_3. Verify that the $[T]_{B_i}$'s are related by the proper conjugation operation. (This last part should be three different checks.)

4.1. **Solution.** This can also be approached using the big theorems. Again, we work with two facts:

- The change of basis from B to B' is just the matrix $[\mathrm{Id}_V]_{B'}^{B}$, where $\mathrm{Id}_V : V \to V$ is the identity mapping.
- The change of basis matrix from B to B' is the inverse of the change of basis matrix from B' to B.

So, using the theorem from class on compositions, we see that the change of basis P_{ij} from B_i to B_j is

$$\begin{aligned}P_{ij} &= [\mathrm{Id}_V]_{B_j}^{B_i} \\ &= [\mathrm{Id}_V]_{B_0}^{B_i}[\mathrm{Id}_V]_{B_j}^{B_0} \\ &= \left([\mathrm{Id}_V]_{B_i}^{B_0}\right)^{-1}[\mathrm{Id}_V]_{B_j}^{B_0} \\ &= P_i^{-1}P_j\end{aligned}$$

This makes it possible to compute that

$$P_{12} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & -2 & 1 \\ -1 & 1 & 0 \end{pmatrix},$$

$$P_{23} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -3 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1/2 & -1/2 \\ -2 & 3/2 & 1/2 \\ 3 & 3/2 & 1/2 \end{pmatrix},$$

and

$$P_{13} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} -3 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -1 & -1 \\ 6 & -2 & -1 \\ -1 & 2 & 1 \end{pmatrix}.$$

Then one can check directly that the following three relationships hold with the matrices as above:

$$\begin{aligned}[T_A]_{B_2} &= P_{12}^{-1}[T_A]_{B_1}P_{12} \\ [T_A]_{B_3} &= P_{23}^{-1}[T_A]_{B_2}P_{23} \\ [T_A]_{B_3} &= P_{13}^{-1}[T_A]_{B_1}P_{13}\end{aligned}$$

I have to admit, I used MATLAB to multiply and invert all the matrices in this assignment. Don't tell the other professors, please, or my mom.

5. Verify the big theorems

Let $x = \begin{pmatrix} 4 & -2 & 7 \end{pmatrix}^t \in \mathbb{R}^3$. Write the coordinate representation of x and $T(x)$ with respect to the three different bases above. Verify that the change of coordinate matrices change coordinates in the proper way. (Three checks for each vector.) Verify that each of the coordinate representations satisfy $[T]_{B_i}[x]_{B_i} = [T(x)]_{B_i}$.

5.1. **Solution.** Again, we'll use the theorems a bit. The coordinate representation of x with respect to B_0 is x itself. So by the coordinate change of basis for points, we see that

$$[x]_{B_1} = P_i^{-1}[x]_{B_0} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ -2 \\ 7 \end{pmatrix} = \begin{pmatrix} -5 \\ -15 \\ 9 \end{pmatrix}$$

and also that

$$[T_A(x)]_{B_1} = P_i^{-1}[T_A(x)]_{B_0} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 22 \\ 7 \end{pmatrix} = \begin{pmatrix} 10 \\ 22 \\ -5 \end{pmatrix},$$

where we can check that

$$[T_A]_{B_1}[x]_{B_1} = \begin{pmatrix} 10 & 8 & 20 \\ 13 & 11 & 28 \\ -5 & -4 & -10 \end{pmatrix} \begin{pmatrix} -5 \\ -15 \\ 9 \end{pmatrix} = \begin{pmatrix} 10 \\ 22 \\ -5 \end{pmatrix} = [T_A(x)]_{B_1}$$

This verifies the big theorem. (Well, this way it really has to verify it, huh? We used the big theorems to write it out.)

You can do the others similarly. So you can double check your calculations, I'll write down the critical parts:

$$[x]_{B_2} = \begin{pmatrix} -5/2 \\ 13/2 \\ 1/2 \end{pmatrix}, \qquad [T(x)]_{B_2} = \begin{pmatrix} 5 \\ 0 \\ 17 \end{pmatrix},$$

and

$$[x]_{B_3} = \begin{pmatrix} -6/5 \\ 2/5 \\ 7 \end{pmatrix}, \qquad [T(x)]_{B_3} = \begin{pmatrix} 17/5 \\ 76/5 \\ -32 \end{pmatrix}.$$

6. On composition

Now consider the map $S : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $S(\begin{pmatrix} a & b & c \end{pmatrix}^t) = \begin{pmatrix} a+b+c & 2a-c \end{pmatrix}^t$. Give $\mathbb{R}^2$ the basis

$$E = \{\begin{pmatrix} 1 & 1 \end{pmatrix}^t, \begin{pmatrix} 1 & -1 \end{pmatrix}^t\}.$$

Compute the matrix representations $[S]_{B_1}^E$ and $[S \circ T]_{B_1}^E$ and verify that

$$[S \circ T]_{B_1}^E = [S]_{B_1}^E [T]_{B_1}.$$

6.1. **Solution.** Okay, enough with the craziness. I'll just write down the two new matrices you need to find.

$$[S]_{B_1}^E = \begin{pmatrix} 2 & 1/2 & 5/2 \\ 1 & 3/2 & 7/2 \end{pmatrix}, \quad [S \circ T]_{B_2}^E = \begin{pmatrix} 14 & 23/2 & 29 \\ 12 & 21/2 & 27 \end{pmatrix}$$

These can be done in the standard way, or with a version of the trick above. Then you need to check that

$$[S]_{B_1}^E [T]_{B_1} = \begin{pmatrix} 2 & 1/2 & 5/2 \\ 1 & 3/2 & 7/2 \end{pmatrix} \begin{pmatrix} 10 & 8 & 20 \\ 13 & 11 & 28 \\ -5 & -4 & -10 \end{pmatrix} = \begin{pmatrix} 14 & 23/2 & 29 \\ 12 & 21/2 & 27 \end{pmatrix} = [S \circ T]_{B_1}^E.$$

`Now, it may seem like I withheld information from you on how to best complete this assignment. This is not really true. The trick I used here only works for matrix maps like` T_A `between coordinate spaces. In general, you have to work all of the definitions and do the basic routine. However, I think that understanding how I did this problem with the trick is a good test of your understanding of the big theorems. I think it is worth studying.`