MATH 211 HOMEWORK ASSIGNMENT 8

FALL 2006, WILLIAMS COLLEGE

ABSTRACT. This assignment has 6 problems on 2 pages. It is due Wednesday, November 21 by 5pm. Don't hesitate to ask for help. This may seem long, but it is important to do this stuff now, and not after a break.

1. A Property of the transpose

Let A and B be square matrices. Show that $(AB)^t = B^t A^t$.

2. Some more about Bases

Let $T : V \to W$ be an invertible linear transformation, and let $B = \{v_1, \dots, v_n\}$ be a basis of V. Show that the set $T(B) = \{T(v_1), \dots, T(v_n)\}$ is a basis of W.

3. Matrix representations

Let $T = T_A : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation determined by the matrix

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 7 & 4 \\ 1 & 4 & 3 \end{pmatrix}.$$

Find the matrix representations of T with respect to each of the following bases

(1) $B_1 = \{u_1 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^t, u_2 = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}^t, u_3 = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}^t\}$,
(2) $B_2 = \{v_1 = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^t, v_2 = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^t, v_3 = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}^t\}$,
(3) $B_3 = \{w_1 = \begin{pmatrix} -3 & 2 & 1 \end{pmatrix}^t, w_2 = \begin{pmatrix} 1 & 1 & 3 \end{pmatrix}^t, w_3 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^t\}$.

4. Changing Basis

Let B_1, B_2, B_3 be as in the last problem. Find the change of basis matrices from B_1 to B_2, from B_2 to B_3 and from B_1 to B_3. Verify that the $[T]_{B_i}$'s are related by the proper conjugation operation. (This last part should be three different checks.)

5. Verify the big theorems

Let $x = \begin{pmatrix} 4 & -2 & 7 \end{pmatrix}^t \in \mathbb{R}^3$. Write the coordinate representation of x and $T(x)$ with respect to the three different bases above. Verify that the change of coordinate matrices change coordinates in the proper way. (Three checks for each vector.) Verify that each of the coordinate representations satisfy $[T]_{B_i}[x]_{B_i} = [T(x)]_{B_i}$.

6. On composition

Now consider the map $S : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $S(\begin{pmatrix} a & b & c \end{pmatrix}^t) = \begin{pmatrix} a+b+c & 2a-c \end{pmatrix}^t$. Give $\mathbb{R}^2$ the basis

$$E = \{\begin{pmatrix} 1 & 1 \end{pmatrix}^t, \begin{pmatrix} 1 & -1 \end{pmatrix}^t\}.$$

Compute the matrix representations $[S]^E_{B_1}$ and $[S \circ T]^E_{B_1}$ and verify that

$$[S \circ T]^E_{B_1} = [S]^E_{B_1}[T]_{B_1}.$$