MATH 211 HOMEWORK 9 SOLUTIONS

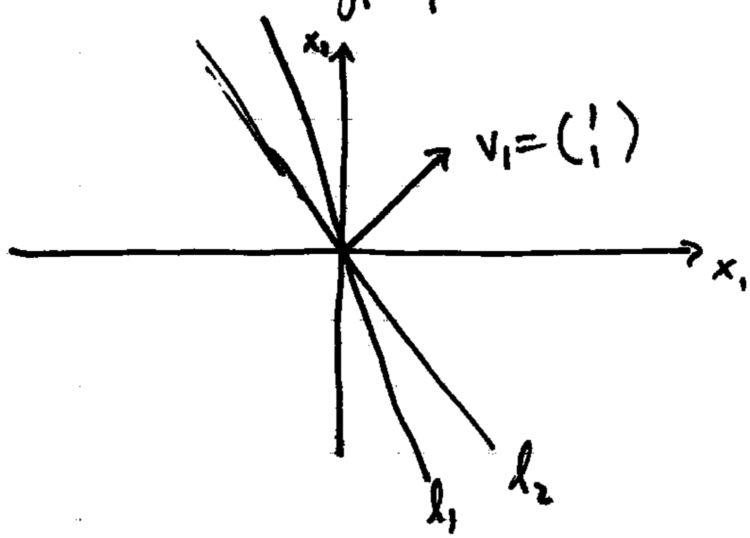
1. The orthogonal set to $v_1 = \begin{pmatrix}1\\1\end{pmatrix}$ is

$$v_1^{\perp} = \left\{ x = \begin{pmatrix}x_1\\x_2\end{pmatrix} \;\middle|\; (1\ \ 1)\begin{pmatrix}2&1\\1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = 0 \right\}$$
$$= \left\{ x = \begin{pmatrix}x_1\\x_2\end{pmatrix} \;\middle|\; 3x_1 + 2x_2 = 0 \right\} = \ell_1$$

For the dot product, the set orthogonal to v_1 is

$$v_1^{\perp} = \left\{ x = \begin{pmatrix}x_1\\x_2\end{pmatrix} \;\middle|\; (1\ \ 1)\begin{pmatrix}x_1\\x_2\end{pmatrix} = 0 \right\}$$
$$= \{ x \mid x_1 + x_2 = 0 \} = \ell_2$$

These are both hyperplanes in $\mathbb{R}^2$:



For $v_2 = \begin{pmatrix}0\\1\end{pmatrix}$, we perform similar computations to find for the new inner product

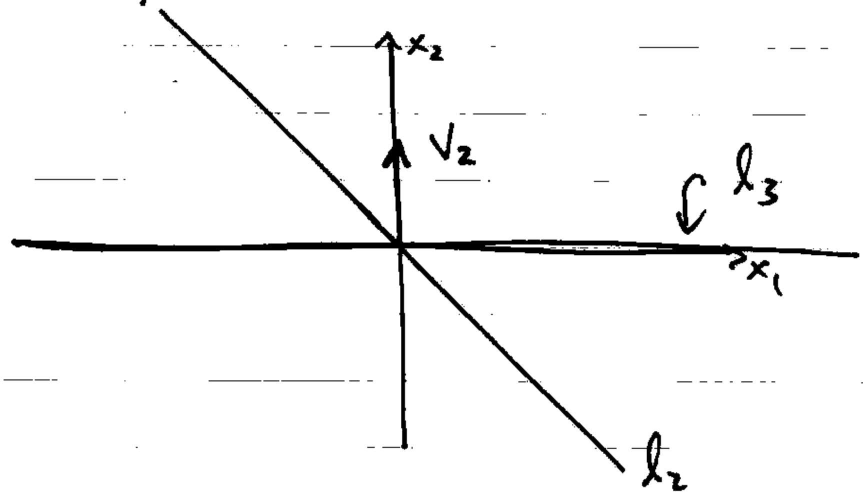
$$v_2^{\perp} = \left\{ x \;\middle|\; (0\ \ 1)\begin{pmatrix}2&1\\1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = 0 \right\} = \ell_2 \text{ from above!}$$

and for the dot product

$$v_2^{\perp} = \left\{ x \;\middle|\; (0\ \ 1)\begin{pmatrix}x_1\\x_2\end{pmatrix} = 0 \right\} = \{ x \mid x_2 = 0 \} \underset{= \ell_3}{=} x\text{-axis!}$$

1 (cont.)

So our picture is



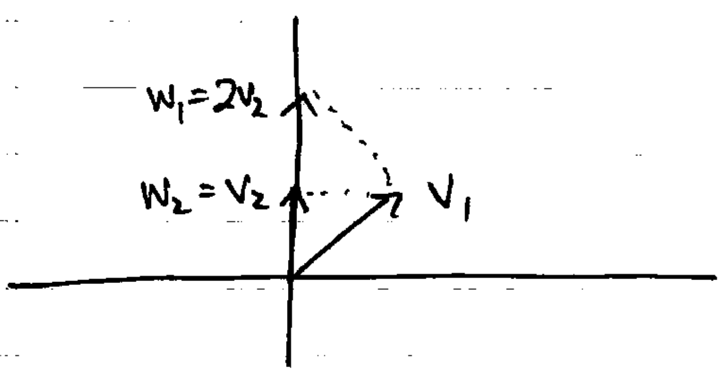
We can see from these pictures that the notions of orthogonal are quite different for these innerproducts.

For the new inner product, we compute

$$W_1 = Pr_{V_2}(V_1) = \frac{\langle V_1, V_2 \rangle}{\langle V_2, V_2 \rangle} V_2 = \frac{2}{1} V_2 = 2V_2$$

For the dot product

$$W_2 = Pr_{V_2}(V_1) = \frac{\langle V_1, V_2 \rangle}{\langle V_2, V_2 \rangle} V_2 = \frac{1}{1} V_2 = V_2$$



These projections are quite different!

② (1) $tr(A) = 5 + (-4) + (-3) = -2$
$tr(B) = 4 + 3 + 1 + 2 = 10$

(2) The matrices X and X^t have the same diagonal entries, so $tr(X) = tr(X^t)$

(3) Let $A = (a_{ij})$ and $B = (b_{ij})$ be $n \times n$ matrices and let $\alpha, \beta \in \mathbb{R}$. Then we have

$$tr(\alpha A + \beta B) = tr(\alpha a_{ij} + \beta b_{ij}) = \sum_{i=1}^{n} (\alpha a_{ii} + \beta b_{ii})$$
$$= \alpha \sum_{i=1}^{n} a_{ii} + \beta \sum_{i=1}^{n} b_{ii} = \alpha\, tr(A) + \beta\, tr(B)$$

Therefore, $tr : M_{nn}(\mathbb{R}) \to \mathbb{R}$ is a linear transformation.

③ To see that the trace form defines an inner product we must check 3 properties:
(1) Symmetry, (2) bilinearity, (3) Positive definite-ness
We take these in order:
(1) By exercise 2, $tr(X) = tr(X^t)$. Note that $A^tB = (B^tA)^t$. Thus
$$\langle A, B\rangle = tr(B^tA) = tr(A^tB) = \langle B, A\rangle.$$
(2) Since tr is a linear transformation, we have
$$\langle \alpha A_1 + \beta A_2, B\rangle = tr(B^t(\alpha A_1 + \beta A_2))$$
$$= tr(\alpha B^tA_1 + \beta B^tA_2)$$
$$= \alpha\, tr(B^tA_1) + \beta\, tr(B^tA_2)$$
$$= \alpha \langle A_1, B\rangle + \beta \langle A_2, B\rangle.$$

3 (cont.)

(3) Let $A=(a_{ij})$. Then $A^t=(b_{ij})$ where $b_{ij}=a_{ji}$. And we get $A^tA=(c_{ij})$ where $c_{ij}=\sum_{k=1}^n b_{ik}a_{kj}$

$$= \sum_{k=1}^n a_{ki}a_{kj}$$

thus $\operatorname{tr}(A^tA)=\sum_{i=1}^n\sum_{k=1}^n a_{ki}^2$.

As a sum of squares it is certainly non-negative. And it equals zero exactly when each $a_{ik}=0$, i.e. when $A=0$

It follows from this last computation that

$$\|A\| = \sqrt{\langle A,A\rangle} = \sqrt{\sum_{i,j} a_{ij}^2}$$

To see that $B=\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$ is an orthonormal basis, we just compute

$$\operatorname{tr}\left(\begin{pmatrix}1&0\\0&0\end{pmatrix}^t\begin{pmatrix}1&0\\0&0\end{pmatrix}\right)=1=\operatorname{tr}\left(\begin{pmatrix}0&1\\0&0\end{pmatrix}^t\begin{pmatrix}0&1\\0&0\end{pmatrix}\right)=\operatorname{tr}\left(\begin{pmatrix}0&0\\1&0\end{pmatrix}^t\begin{pmatrix}0&0\\1&0\end{pmatrix}\right)$$
$$=\operatorname{tr}\left(\begin{pmatrix}0&0\\0&1\end{pmatrix}^t\begin{pmatrix}0&0\\0&1\end{pmatrix}\right)$$

so all vectors have norm one, and

$$\operatorname{tr}\left(\begin{pmatrix}1&0\\0&0\end{pmatrix}^t\begin{pmatrix}0&1\\0&0\end{pmatrix}\right)=0=\operatorname{tr}\left(\begin{pmatrix}1&0\\0&0\end{pmatrix}^t\begin{pmatrix}0&0\\1&0\end{pmatrix}\right)=\operatorname{tr}\left(\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix}\right)$$
$$=\operatorname{tr}\left(\begin{pmatrix}0&1\\0&0\end{pmatrix}^t\begin{pmatrix}0&0\\1&0\end{pmatrix}\right)=\operatorname{tr}\left(\begin{pmatrix}0&1\\0&0\end{pmatrix}^t\begin{pmatrix}0&0\\0&1\end{pmatrix}\right)$$
$$=\operatorname{tr}\left(\begin{pmatrix}0&0\\1&0\end{pmatrix}^t\begin{pmatrix}0&0\\0&1\end{pmatrix}\right)$$

so the set B is orthogonal.

4 We begin with the vectors

$$v_1 = \begin{pmatrix}1\\0\\1\\2\end{pmatrix} \quad v_2 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix} \quad v_3 = \begin{pmatrix}1\\1\\2\\4\end{pmatrix} \quad v_4 = \begin{pmatrix}1\\2\\-4\\-3\end{pmatrix}$$

first we make an orthogonal set

$$w_1 = v_1 = \begin{pmatrix}1\\0\\1\\2\end{pmatrix}, \quad \langle w_1, w_1\rangle = 1+0+1+2 = 6$$

$$\|w_1\| = \sqrt{6}$$

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1$$

$$= \begin{pmatrix}1\\1\\1\\1\end{pmatrix} - \frac{4}{6}\begin{pmatrix}1\\0\\1\\2\end{pmatrix} = \begin{pmatrix}1/3\\1\\1/3\\-1/3\end{pmatrix}, \quad \langle w_2, w_2\rangle = \frac{1}{9}+1+\frac{1}{9}+\frac{1}{9} = \frac{12}{9} = \frac{4}{3}$$

$$\|w_2\| = \frac{2}{\sqrt{3}}$$

$$w_3 = v_3 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1$$

$$= \begin{pmatrix}1\\1\\2\\4\end{pmatrix} - \frac{2/3}{4/3}\begin{pmatrix}1/3\\1\\1/3\\-1/3\end{pmatrix} - \frac{11}{6}\begin{pmatrix}1\\0\\1\\2\end{pmatrix} = \begin{pmatrix}1\\1\\2\\4\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1/3\\1\\1/3\\-1/3\end{pmatrix} - \frac{11}{6}\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$$

$$= \begin{pmatrix}-1\\1/2\\0\\1/2\end{pmatrix}, \quad \langle w_3, w_3\rangle = 1+\frac{1}{4}+0+\frac{1}{4} = \frac{6}{4} = \frac{3}{2}$$

$$\|w_3\| = \sqrt{3/2}$$

$$w_4 = v_4 - \frac{\langle v_4, w_3\rangle}{\langle w_3, w_3\rangle} w_3 - \frac{\langle v_4, w_2\rangle}{\langle w_2, w_2\rangle} w_2 - \frac{\langle v_4, w_1\rangle}{\langle w_1, w_1\rangle} w_1$$

$$= \begin{pmatrix}1\\2\\-4\\-3\end{pmatrix} - \frac{(-3/2)}{3/2}\begin{pmatrix}-1\\1/2\\0\\1/2\end{pmatrix} - \frac{2}{4/3}\begin{pmatrix}1/3\\1\\1/3\\-1/3\end{pmatrix} - \frac{(-9)}{6}\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$$

$$= \begin{pmatrix}1\\2\\-4\\-3\end{pmatrix} + \begin{pmatrix}-1\\1/2\\0\\1/2\end{pmatrix} - \frac{3}{2}\begin{pmatrix}1/3\\1\\1/3\\-1/3\end{pmatrix} + \frac{3}{2}\begin{pmatrix}1\\0\\1\\2\end{pmatrix} = \begin{pmatrix}1\\+1\\-3\\1\end{pmatrix} \quad \text{(oops)}$$

4 (cont.)

To get an orthonormal basis, we divide these vectors by their lengths. The only one yet to compute is

$\langle w_4, w_4 \rangle = 1+1+9+1 = 12 \quad , \quad \|w_4\| = \sqrt{12}$

so our orthonormal basis is

$$u_1 = \begin{pmatrix} 1/\sqrt{6} \\ 0 \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}, \quad u_2 = \begin{pmatrix} \frac{1}{2\sqrt{3}} \\ \sqrt{3}/2 \\ \frac{1}{2\sqrt{3}} \\ \frac{-1}{2\sqrt{3}} \end{pmatrix}$$

$$u_3 = \begin{pmatrix} -\sqrt{\frac{2}{3}} \\ \frac{1}{2}\sqrt{\frac{2}{3}} \\ 0 \\ \frac{1}{2}\sqrt{\frac{2}{3}} \end{pmatrix}, \quad u_4 = \begin{pmatrix} 1/\sqrt{12} \\ 1/\sqrt{12} \\ -3/\sqrt{12} \\ 1/\sqrt{12} \end{pmatrix}.$$

5 To do this problem: One way is to pick any basis $v_1, \dots, v_n$ of $\mathbb{R}^n$, then apply Gramm-Schmidt to produce an orthonormal basis of $\mathbb{R}^n$. By the theorem from class on Monday, the matrix with these vectors as its columns is orthogonal!

There are lots of possible answers, and they can be checked by computing that $X^t X = I$.

For example, a 4x4 matrix can be made by the last problem.

Cleaned up $\longrightarrow$ $\begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{12} & -2/\sqrt{6} & 1/\sqrt{12} \\ 0 & 3/\sqrt{12} & 1/\sqrt{6} & 1/\sqrt{12} \\ 1/\sqrt{6} & 1/\sqrt{12} & 0 & -3/\sqrt{12} \\ 1/\sqrt{6} & -1/\sqrt{12} & 1/\sqrt{6} & 1/\sqrt{12} \end{pmatrix}$ is orthogonal.

(6) (1) Let $B = \{v_1, \dots, v_n\}$ be a basis of V.
We apply Gramm-Schmidt to produce an orthogonal basis $\{w_1, \dots, w_n\} = B'$.

$$
\begin{aligned}
w_1 &= v_1 \\
w_2 &= v_2 - c_{21} w_1 \\
w_3 &= v_3 - c_{32} w_2 - c_{31} w_1 \\
&\vdots \\
w_n &= v_n - c_{n\,n-1} w_{n-1} - \cdots - c_{n1} w_1
\end{aligned}
$$

the c_{ij}'s are ~~co~~ scalars...

Two ways from here:

$\boxed{1}$ bring the w_i terms on the right over to the left! Obtain

$$
\begin{aligned}
v_1 &= w_1 \\
v_2 &= c_{21} w_1 + w_2 \\
v_3 &= c_{31} w_1 + c_{32} w_2 + w_3 \\
&\vdots \\
v_n &= c_{n1} w_1 + c_{n2} w_2 + \cdots + w_1
\end{aligned}
$$

The change of basis matrix from B' to B is the transpose of this array. Call it Q. Clearly, Q is upper triangular.

Now, the change of basis matrix from B to B' is $P = Q^{-1}$, and the inverse of an upper triangular matrix is upper triangular (can you prove that?) so we're done

6 (cont.)

$\boxed{2}$ Use the upper equations to substitute into lower equations in the Gramm–Schmidt equations to see that for some other scalars a_{ij}

$$\begin{aligned} w_1 &= v_1 \\ w_2 &= a_{21}v_1 + v_2 \\ w_3 &= a_{31}v_1 + a_{32}v_2 + v_3 \\ &\vdots \\ w_n &= a_{n1}v_1 + \cdots + v_n \end{aligned}$$

This exhibits the change of basis matrix from B to B' as the transpose P of this array, which is clearly upper triangular.

(2) First note that in (1) we could have also normalized our vectors w_i to have an orthonormal basis, and though this changes the actual coefficients it doesn't change the triangularity. So we may assume (1) with B' an onb.

Let $A = (v_1 \cdots v_n)$ be invertible. Then the columns $v_1, \ldots, v_n$ are a basis. By (1), there is an onb $w_1, \ldots, w_n$ of $\mathbb{R}^n$ such that the change of basis P from v's to w's is upper triangular. Recall, $v_i = Pw_i$ by the change of basis theorem. Now we have $A = (v_1 \cdots v_n) = (Pw_1 \cdots Pw_n) = P(w_1 \cdots w_n) = PK$ where $K = (w_1 \cdots w_n)$ has orthonormal columns and is hence orthogonal.