HOMEWORK ASSIGNMENT # 9

MATH 211, FALL 2006, WILLIAMS COLLEGE

Abstract. This assignment has six problems on two pages. It is due on Tuesday, December by 5pm. Good Luck!

1. The geometry of an inner product

Consider the inner product on $\mathbb{R}^2$ given by the formula

$$\langle x, y\rangle = x^t \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} y.$$

(There is no need to check that this is an inner product–take it for granted.) Find, describe and draw the set of vectors which are orthogonal to $v_1 = \begin{pmatrix} 1 & 1 \end{pmatrix}^t$ with respect to this inner product (and be sure to include v_1 on the same set of axes). Do the same process for $v_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}^t$.

How do these sets compare to the same construction with the standard dot product $\langle x, y\rangle = x^t y$?

Compute the orthogonal projection of v_1 onto v_2 with respect to the first inner product and draw the corresponding picture. For comparison, do the same with the dot product.

2. The trace

Let X be a square matrix. (That is, $X \in M_{nn}(\mathbb{R})$.) The *trace* of X, denoted $\mathrm{tr}(X)$ is the sum of the diagonal entries of X.

(1) Compute the trace of the following matrices:

$$A = \begin{pmatrix} 5 & 3 & 1 \\ 4 & -4 & 3 \\ 2 & 1 & -3 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 4 & 1 & 1000 & 2 \\ 7 & 3 & -1000 & -1 \\ 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 2 \end{pmatrix}.$$

(2) Show that $\mathrm{tr}(X) = \mathrm{tr}(X^t)$.

(3) Show that $\mathrm{tr} : M_{nn}(\mathbb{R}) \to \mathbb{R}$ is a linear transformation.

3. The trace form

Let $V = M_{mn}(\mathbb{R})$ be the vector space of all real $m \times n$ matrices. Show that the function

$$\langle A, B\rangle = \mathrm{tr}(B^t A)$$

defines an inner product on V. This is called the *trace form*. What is the associated norm? That is, for an $m \times n$ matrix A, give a formula for $\|A\|$. Show that the standard basis

$$B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is an orthonormal basis with respect to the trace form on $M_{22}(\mathbb{R})$.

4. Gramm-Schmid

Apply the Gramm-Schmid orthogonalization process to the following set to produce an orthonormal basis of $\mathbb{R}^4$.

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 4 \end{pmatrix}, v_4 = \begin{pmatrix} 1 \\ 2 \\ -4 \\ -3 \end{pmatrix}.$$

5. Orthogonal matrices

Find three different examples of orthogonal matrices, one each that is square of size 2, 3 and 4. Prove that your matrices are orthogonal. (Don't use examples from class.)

6. A matrix decomposition theorem

(1) Show that for any basis B of a finite dimensional vector space V there is an orthogonal basis B' of V such that the change of basis matrix from B to B' is upper triangular.
(2) Use the above to prove that any invertible matrix can be written as the product of an orthogonal matrix and an upper triangular matrix.