SECOND EXAM SOLUTIONS

MATH 211, WILLIAMS COLLEGE, FALL 2006

Abstract. These are the instructor's solutions for the second exam. For statements of the problems, see the posted copy of the exam.

1. Problem One

To see if W_1 and W_2 are the same, we apply the row space algorithm to the given spanning sets to find the "canonical" bases. For W_1 and W_2, respectively, this yields

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 0 & 3 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, we see that $W_1 = \text{span}\{\begin{pmatrix} 1 & 2 & 1 \end{pmatrix}^t, \begin{pmatrix} 0 & 3 & 2 \end{pmatrix}^t\} = W_2$, hence the two spaces are the same.

For the second part, recall that we can find a basis consisting of a subset of a given spanning set by applying the column space algorithm. The relevant matrix of columns (and its row echelon form) is

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 1 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since the first and second columns are the ones with pivots, we find that $\{v_1, v_2\}$ is a basis of W_2. There are other ways to do this second part, and it means that any pair of the vectors v_1, v_2, v_3 forms a basis.

2. Problem Two

Suppose that the hyperplanes are given by equations

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5x_5 + a_6x_6 = c_1$$
$$b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 + b_5x_5 + b_6x_6 = c_2.$$

Then the intersection is the same as the solution set to the matrix equation $Ax = c$ where

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{pmatrix}, \quad \text{and} \quad c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

By the rank-nullity theorem applied to the matrix multiplication mapping $T_A : \mathbb{R}^6 \to \mathbb{R}^2$, we see that $6 = \dim \mathbb{R}^6 = \text{rank}(A) + \text{nullity}(A)$. But the rank of A is the number of linearly independent rows of A, and thus can't be any larger than 2. So we conclude that $\text{nullity}(A) \geq 4$. Hence, the null space of A is at least 4-dimensional. Finally, we know that the solution set to $Ax = c$ is a translate of this null space, and hence has the same dimension.

3. Problem Three

To see that $S_2(\mathbb{R})$ is a subspace, we must show that it contains the zero vector and is closed under linear combinations. First, the zero vector in $M_2(\mathbb{R})$ is the matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. This is certainly symmetric, and hence an element of $S_2(\mathbb{R})$. Now let A and B be a pair of symmetric matrices, and a, b a pair of real numbers. We must show that $aA + bB$ is symmetric. One can do this directly by computing with a pair of generic 2×2-matrices, or note that $(A + B)^t = A^t + B^t$ so that

$$(aA + bB)^t = aA^t + bB^t = aA + bB.$$

Thus, $aA + bB$ is symmetric.

Now we show that T is a linear transformation. We work with the linear combination above and compute:

$$\begin{aligned} T(aA + bB) &= \frac{1}{2}\left((aA + bB) + (aA + bB)^t\right) = \frac{1}{2}(aA + bB + aA^t + bB^t) \\ &= \frac{a}{2}(A + A^t) + \frac{b}{2}(B + B^t) = aT(A) + bT(B) \end{aligned}$$

Thus, T is a linear transformation.

The kernel of T is the set of matrices A such that $0 = T(A) = \frac{1}{2}(A + A^t)$. That is, the matrices which have $A^t = -A$. These are the *skew-symmetric* matrices. The subspace of these is spanned by the single matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. So we see that $\text{nullity}(T) = 1$. (I've shortened this by omitting the computation that this is a basis, but it is pretty straightforward.)

The image of T is the entire set $S_2(\mathbb{R})$ of symmetric matrices. To see this, realize that if A is a symmetric matrix, then $T(A) = \frac{1}{2}(A + A^t) = \frac{1}{2}(A + A) = A$. The set $S_2(\mathbb{R})$ has a basis consisting of the three matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence $\text{rank}(T) = 3$. (Similar to the above step, it is not hard to check that this set is a basis.)

This allows us to verify the rank-nullity theorem as follows:

$$\dim(M_2(\mathbb{R})) = 4 = 3 + 1 = \text{rank}(T) + \text{nullity}(T).$$

Note that this problem requires playing with matrices a little bit. In particular, if you are stuck, you need to be able to see that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

4. Problem Four

Let v, w be two vectors in V and let a, b be a pair of scalars. To show that $S \circ T$ is a linear transformation, we compute that

$$\begin{aligned}(S \circ T)(av + bw) &= S(T(av + bw)) = S(aT(v) + bT(w)) \\ &= aS(T(v)) + bS(T(w)) = a(S \circ T)(v) + b(S \circ T)(w).\end{aligned}$$

So $(S \circ T)$ is a linear transformation.

As for the nullity part of the problem: It is not always true that $\text{nullity}(S \circ T) = \text{nullity}(S) + \text{nullity}(T)$. For an example, consider the matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We let $S = T_B : \mathbb{R}^3 \to \mathbb{R}^3$ and $T = T_A : \mathbb{R}^3 \to \mathbb{R}^3$. Then $\text{nullity}(S) = 1$ and $\text{nullity}(T) = 2$, but $\text{nullity}(S \circ T) = 2 \neq 1 + 2$, since $S \circ T$ is left multiplication by

$$BA = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = B.$$