Final Exam

Math 211, Williams College

December 15, 2006

Print your name here: ________________________________

Directions: This Exam has 6 questions on 8 pages, including this cover. Be sure to show all of your work. You have two and a half hours to complete this exam. Please sign the honor code pledge below.

Pledge: On my honor, I have neither given nor received any aid on this examination.

Write pledge and sign name here:

Problem	*Possible*	*Score*
1	2	
2	4	
3	9	
4	7	
5	4	
6	4	
Total	30	

1. Examples

Give two examples of matrices which are not diagonalizable over the real numbers. Your two examples should fail for different reasons.

2. A nice basis

Find an orthonormal basis of $\mathbb{R}^3$ (under the dot product) that contains the vector $x = \begin{pmatrix} -1/3 & 2/3 & 2/3 \end{pmatrix}^t$.

3. Matrix representations

Let

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, v_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}.$$

Note that $B = \{u_1, u_2, u_3\}$ and $B' = \{v_1, v_2, v_3\}$ are bases of $\mathbb{R}^3$.

(1) Compute the change of basis matrix P from B to B' and the change of basis matrix Q from B' to B.

(2) Are P and Q related? Are either of P or Q orthogonal?

(Problem 3 continued)

(3) Write the matrix representation with respect to B of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 4y - z \\ 2x + z \\ 3y + 5z \end{pmatrix}.$$

(4) Write a product of three matrices which computes the matrix representation of T with respect to B' using matrices you have already computed. (You need not evaluate this product.)

4. Diagonalization

Consider the linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x) = Ax$ where A is the matrix

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & -4 \\ 1 & 1 & 1 \end{pmatrix}.$$

(1) Is this linear transformation diagonalizable? If so, write out a basis of eigenvectors and their corresponding eigenvalues.

(Problem 4 continued)

(2) Write down a product of three matrices which computes the matrix which represents the linear transformation T^{100} (the composition of one hundred copies of T) with respect to the canonical basis of $\mathbb{R}^3$. (You don't need to actually multiply them.)

5. A Proof

Choose one of the following two statements and give a detailed sketch of how to prove it. (That is, you don't have to check every detail, but you must indicate you know what the details are.) Indicate clearly which one you want graded.

Option A: Let V and W be finite dimensional real vector spaces. Prove that V and W are isomorphic if and only if $\dim V = \dim W$.

Option B: Let A and B be two $m \times n$ real matrices with real entries. Show that $\text{null}(A) = \text{null}(B)$ if and only there is a sequence of elementary matrices $E_1, \ldots, E_n$ such that $B = E_n \ldots E_1 A$.

6. The Spectral theorem

(1) Suppose that A is a symmetric real matrix. What does it mean for A to be *positive definite*?

(2) State the *Spectral Theorem* as given in class.

(3) Based on the above, think up a test (other than the definition) for when a symmetric matrix A is positive definite.