Final Exam

Math 251, Williams College

December 16, 2006

Directions: This Exam has 6 questions on 8 pages. Be sure to show all of your work and explain yourself carefully. You have two and one half hours to complete this exam. Please sign the honor code pledge below.

Print your name here: ________________________________

Pledge: On my honor, I have neither given nor received any aid on this examination.

Write pledge and sign name here:

Problem	*Possible*	*Score*
1	6	
2	3	
3	4	
4	4	
5	5	
6	8	
Total	30	

1. Recurrence Relations

(1) In climbing up a staircase, an ordinary step covers at least one stair and at most two stairs. Let A_n be the number of ways to climb up a staircase of n stairs making only ordinary steps. Find a recurrence relation for A_n.

(2) Solve the following recurrence relation.

$$\begin{cases} b_n &= b_{n-1} + 2n + 1 \\ b_0 &= 1 \end{cases}$$

2. Degrees

(1) Prove the *First Theorem of Graph Theory*: The sum of the degrees of all vertices is even.

(2) Can the numbers $5, 4, 3, 4, 2$ be the degree sequence of a graph? If so, give an example. If not, say why not.

3. Special Circuits

Draw examples of <u>connected</u> graphs with the following properties:

(1) five vertices, Eulerian but not Hamiltonian,

(2) four vertices, Hamiltonian but not Eulerian,

(3) four vertices, both Hamiltonian and Eulerian,

(4) four vertices, neither Hamiltonian nor Eularian.

4. Trees

(1) Define what it means for a graph to be a *tree*.

(2) Show that a connected graph in which every vertex is even must contain a cycle.

5. Planarity

(1) State Euler's formula for plane graphs.

(2) Use induction to prove Euler's formula for plane graphs.

6. Mail Delivery

Consider the graph G, below, as a street map with lengths of streets (in hundreds of yards) given by the labels. You are hired to work as a postal carrier in this town. The post office is at vertex I. (You should use the blank page which follows for your work.)

(1) Find the shortest path in G from vertex A to vertex Z. Describe this path as a sequence of vertices.

(2) Which streets should you use to double back in order to keep the total amount of walking to a minimum? Describe them as pairs of vertices.

(3) What path you should take to cover your entire route with the least amount of walking? Describe it as a sequence of vertices.

(4) To keep you safe in the dark winter months, the town has agreed to put up an emergency phone at each corner. The system is to have it own dedicated set of phone lines to the post office. Along which roads should the lines be place to minimize the amount of wire used? Shade these roads on the graph below.

Use this page for your work on problem 6.