HOMEWORK ASSIGNMENT # 1

MATH 251, FALL 2006, WILLIAMS COLLEGE

1. Problem One

Construct the truth table of $q \longleftrightarrow ((\neg p) \vee (\neg q))$.

1.1. Solution.

To fit the table on the page, we use x to denote $(\neg p) \vee (\neg q)$.

p	q	$\neg p$	$\neg q$	$(\neg p) \vee (\neg q) = x$	$q \longrightarrow x$	$x \longrightarrow q$	$q \longleftrightarrow x$
T	T	F	F	F	F	T	F
T	F	F	T	T	T	F	F
F	T	T	F	T	T	T	T
F	F	T	T	T	T	F	F

2. Problem Two

Suppose that p and r are false statements and that q and s are true statements. Find the truth values of the following statements.

(1) $(p \longrightarrow q) \longrightarrow r$
(2) $(s \longrightarrow (p \wedge (\neg r))) \wedge ((p \longrightarrow (r \vee q)) \wedge s)$

2.1. Solution.

We construct just the relevant row of the truth table. For part (1):

p	q	r	s	$p \longrightarrow q$	$(p \longrightarrow q) \longrightarrow r$
F	T	F	T	T	F

We continue in part (2) with more of the row we need. Pay attention to the new symbols used to simplify things.

$\neg r$	$y = (p \wedge (\neg r))$	$s \longrightarrow y$	$r \vee q$	$z = (p \longrightarrow (r \vee q))$
T	F	F	T	T

and we continue

$z \wedge s$	$(s \longrightarrow y) \wedge (z \wedge s)$
T	F

So we see that both statements are **false**.

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3. Problem Three

Consider the statement $\mathcal{A}$: "If n is an integer, then $\dfrac{n}{n+1}$ is not an integer."

(1) Is $\mathcal{A}$ true or false? Either prove it is true, or give a counterexample.
(2) Write the converse, contrapositive and negation of $\mathcal{A}$.

3.1. Solution.

In part one, the statement is false. My counterexample is $n = 0$. That is note the truth of the statements: "$n = 0$ is an integer" (so the hypothesis is true), and "$\dfrac{0}{1} = 0$ is an integer" (so the conclusion is false).

For part two: The converse of $\mathcal{A}$ is "If $\dfrac{n}{n+1}$ is not an integer, then n is an integer." The contrapositive is "If $\dfrac{n}{n+1}$ is an integer, then n is not an integer." Finally, the negation is "n is an integer and $\dfrac{n}{n+1}$ is an integer."

Notice that our counterexample is a number which makes the negation true. That is what a counterexample should be.

4. Problem Four

Given the premises $p \longrightarrow (\neg r)$ and $r \vee q$, write down a valid conclusion that involves p and q only and is not a tautology, or show that no such conclusion is possible.

4.1. Solution.

I constructed a big truth table to figure out what was going on. Here it is, in all its glory.

p	q	r	$\neg r$	$a = (p \longrightarrow (\neg r))$	$b = (r \vee q)$	$a \wedge b$	x	$a \wedge b \longrightarrow x$
T	T	T	F	F	T	F	1	T
T	T	F	T	T	T	T	1	T
T	F	T	F	F	T	F	2	T
T	F	F	T	T	F	F	2	T
F	T	T	F	T	T	T	3	T
F	T	F	T	T	T	T	3	T
F	F	T	F	T	T	T	4	T
F	F	F	T	T	F	F	4	T

We need to find a statement x involving only p and q (but possibly being some horrible compound statement), which is not a tautology and makes the statement

$$(p \longrightarrow (\neg r)) \wedge (r \vee q) \longrightarrow x$$

into a tautology. (Recall that this second requirement is the way to recognize a valid argument.)

Now, look at the table. First, notice that the truth value of x in the first two rows must agree. This is because these two rows are the same from the point of view of p and q. We label the value of these rows as 1.

The same can be said about the other pairs of rows, and we label their values as 2, 3 and 4.

If we consult the standard truth table for an implication, we see that there is no choice but to put T as the value of 1, 3 and 4. This is because the conjunction of the hypotheses is true in all of these rows.

In order to avoid x being a tautology, we have to have value 2 equal to F.

So, if we now focus just on the atomic statements p and q and the desired x we get a table that looks like the following.

p	q	x
T	T	T
T	F	F
F	T	T
F	F	T

Fortunately, we can recognize this as the table of $p \longrightarrow q$. So the simplest answer is the statement $p \longrightarrow q$.

5. Problem Five

Decide if the following arguments are valid.

(1)
If I stay up late at night, then I will be tired in the morning.
I stayed up late last night.
―――
I am tired this morning.

(2)
If I stay up late at night, then I will be tired in the morning.
I am not tired this morning.
―――
I stayed up late last night.

(3)
If I stay up late at night, then I will be tired in the morning.
I am not tired this morning.
―――
I did not stay up late last night.

(4)
If I stay up late at night, then I will be tired in the morning.
I did not stay up late last night.
―――
I am not tired this morning.

(5)
If I like mathematics, then I will study.
I will not study.
Either I like mathematics or I like football.
―――
I like football.

(6)
Either I study or I like football.
If I like football, then I like mathematics.
―――
If I don't study, then I like mathematics.

5.1. **Solution.** Argument (2) is invalid: you might have gone to bed early just to be well rested. Argument (4) is invalid: A person can be tired for other reasons, like illness.

All the other arguments are valid. It is possible to test all of these using truth tables, but I can't bear to typeset another one after the last problem.