HOMEWORK ASSIGNMENT # 2

MATH 251, FALL 2006, WILLIAMS COLLEGE

ABSTRACT. These are the instructor solutions to homework 2.

1. PROBLEM ONE

Using Venn diagrams, investigate whether the following statements are true or false.

(1) $A \oplus (B \cap C) = (A \oplus B) \cap (A \oplus C)$
(2) $A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C)$
(3) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$
(4) $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$
(5) $A \cup (B \oplus C) = (A \cup B) \oplus (A \cup C)$

1.1. **Solution.** Items 1, 2 and 5 are false, 3 and 4 are true. I'll sketch how I did item 4. I'll leave the rest to you, as they follow by the same method.

Make a general Venn diagram of sets A, B, C with all possible intersections. This cuts the plane into 8 regions. I'll label them 1 through 8 in just the way I did in class. (So look at your notes!)

Now notice that $B \oplus C = 2 \cup 4 \cup 5 \cup 7$, so that $A \cap (B \oplus C) = 2 \cup 4$. Similarly, we note that $A \cap B = 2 \cup 3$ and $A \cap C = 3 \cup 4$ so that $(A \cap B) \oplus (A \cap C) = 2 \cup 4$. Thus the statement is true.

Date: September 26, 2006.

2. Problem Two

This problem has two parts.

(1) List all the sets in the power set of the following sets:
  (a) $\{a, b\}$
  (b) $\{a, b, c\}$
  (c) $\{\emptyset, 0, \{0\}\}$
(2) List all partitions of the following sets:
  (a) $\{a\}$
  (b) $\{a, b\}$
  (c) $\{a, b, c\}$

2.1. **Solution.** First, part one. (A real surprise, huh?)

(1) $\mathcal{P}(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.
(2) $\mathcal{P}(\{a, b, c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.
(3) $\mathcal{P}(\{\emptyset, 0, \{0\}\}) = \{\emptyset, \{\emptyset\}, \{0\}, \{\{0\}\}, \{\emptyset, 0\}, \{\emptyset, \{0\}\}, \{0, \{0\}\}, \{\emptyset, 0, \{o\}\}\}$.

And now for something completely different:

(1) The only partition of $\{a\}$ is the partition into one single set $(\{a\})$.
(2) This set has two partitions. The first is $(\{a\}, \{b\})$. The second is $(\{a, b\})$.
(3) This set has five partitions. They are
  - $(\{a\}, \{b\}, \{c\})$
  - $(\{a\}, \{b, c\})$
  - $(\{a, b\}, \{c\})$
  - $(\{a, c\}, \{b\})$
  - $(\{a, b, c\})$

3. Problem Three

Let $X = \{a, b, c\}$ and S be the partial order defined on the power set $\mathcal{P}(X)$ given by $S = \{(A, B) \mid A \subseteq B\}$.

(1) List the elements of S.
(2) Draw the Hasse diagram of S.

3.1. **Solution.** S has 27 elements. It is an unwieldy list.

$$
\begin{aligned}
&(\emptyset, \emptyset), (\emptyset, \{a\}), (\emptyset, \{b\}), (\emptyset, \{c\}),\\
&(\emptyset, \{a, b\}), (\emptyset, \{a, c\}), (\emptyset, \{b, c\}), (\emptyset, \{a, b, c\}),\\
&(\{a\}, \{a\}), (\{a\}, \{a, b\}), (\{a\}, \{a, c\}), (\{a\}, \{a, b, c\}),\\
&(\{b\}, \{b\}), (\{b\}, \{a, b\}), (\{b\}, \{b, c\}), (\{b\}, \{a, b, c\}),\\
&(\{c\}, \{c\}), (\{c\}, \{a, c\}), (\{c\}, \{b, c\}), (\{c\}, \{a, b, c\}),\\
&(\{a, b\}, \{a, b\}), (\{a, b\}, \{a, b, c\}), (\{b, c\}, \{b, c\}), (\{b, c\}, \{a, b, c\}),\\
&(\{a, c\}, \{a, c\}), (\{a, c\}, \{a, b, c\}), (\{a, b, c\}, \{a, b, c\})
\end{aligned}
$$

I'll skip the diagram. It has eight nodes. The empty set at the bottom is connected to each of the three singleton sets on the next row. These are each connected to two of the three two-element sets (the third row). Each of the two-element sets is connected to the total set $\{a, b, c\}$, which is all alone on the top row.

4. Problem Four

Let $A = \{a, b, c, d\}$. Draw the digraph of each relation on A given below and decide if the relation is reflexive, symmetric, transitive or antisymmetric.

(1) $R = \{(b, b), (b, c), (b, d), (c, b), (c, c), (c, d)\}$
(2) $R = \{(a, b), (b, a)\}$
(3) $R = \{(a, a), (b, b), (c, c), (d, d)\}$
(4) $R = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a)\}$
(5) $R = \{(a, c), (a, d), (b, c), (b, d), (c, a), (c, d)\}$
(6) $R = \{(a, b), (b, c), (c, d)\}$

4.1. **Solution.** I'll omit the pictures. The first relation is transitive only. The second is only symmetric. The third has all four properties. The fourth is reflexive, transitive and symmetric, but not antisymmetric. The fifth has none of the four properties. The sixth is only anti-symmetric. (think proof by contradiction for this last statement.

5. Problem Five

Let $A = \{2, 3, 4, 6, 8, 12, 16, 24\}$, and let S be the partial order on S defined by $S = \{(a, b) \mid a \text{ divides } b\}$.

(1) Find the minimal elements in A.
(2) Find the maximal elements in A.
(3) Find the upper bounds of the set $B = \{4, 6, 12\}$.
(4) Draw the Hasse diagram of this partial order.

5.1. **Solution.** The minimal elements of A are 2 and 3. The maximal elements are 16 and 24. The least upper bound of B is 12. And the Hasse diagram is omitted.