HOMEWORK ASSIGNMENT # 3 SOLUTIONS

MATH 251, FALL 2006, WILLIAMS COLLEGE

ABSTRACT. These are the instructor's solutions.

1. PROBLEM ONE: ON INJECTIONS

Let $f : A \to B$ and $g : B \to C$ be functions.

(1) Prove that if f and g are injective, then $g \circ f$ is injective.
(2) Suppose that $g \circ f$ is injective. Is it necessarily true that f is injective? Or that g is injective? Give a proof or counterexample to back up your answers.

1.1. **Solution:**

(1) Let x and y be elements of A such that $g \circ f(x) = g \circ f(y) \in C$. Since g is injective, $f(x)$ and $f(y)$ must be the same element of B. But then since f is injective, x and y must be the same element of A. Hence $g \circ f$ is injective.
(2) If $g \circ f$ is injective, then f must be injective, but g need not be. To see that g is not necessarily injective, consider the following example. Let $f : [0, 1] \to [-1, 1]$ be $f(x) = x^2$, and $g : [-1, 1] \to [0, 1]$ be $g(x) = |x|$.

    To prove that f must be injective, we work the contrapositive route. Suppose that f is not injective. Then there exist points x, y such that $f(x) = f(y)$ but $x \neq y$. Then we see that $g \circ f(x) = g(f(x)) = g(f(y)) = g \circ f(y)$, too. Thus $g \circ f$ is not injective.

2. PROBLEM TWO: ON SURJECTIONS

Let $f : A \to B$ and $g : B \to C$ be functions.

(1) Prove that if f and g are surjective, then $g \circ f$ is surjective.
(2) Suppose that $g \circ f$ is surjective. Is it necessarily true that f is surjective? Or that g is surjective? Give a proof or counterexample to back up your answers.

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2.1. **Solution:**

(1) Let x be an element of C. We must produce an element a of A such that $g \circ f(a) = c$.

Since g is surjective, there exists a point $b \in B$ such that $g(b) = c$. Since f is surjective, there exists a point $a \in A$ such that $f(a) = b$.

But now $g \circ f(a) = g(f(a)) = g(b) = c$. So we are done.

(2) If $g \circ f$ is surjective, then so must be g, but not f. To see that f need not be surjective, consider the example of

$$f : [0, 1] \to [0, 1], g : [0, 1] \to [0, 1]$$

defined by $f(x) = x/2$ and

$$g(x) = \begin{cases} 2x, & 0 \leq x \leq 1/2, \\ 1, & 1/2 < x \leq 1. \end{cases}$$

We now prove that g must be surjective (by the contrapositive). Suppose that g is not surjective. Then there is a point $c \in C$ such that $g(B)$ does not contain c. But then $(g \circ f)(A) \subseteq g(B)$ does not contain c, and $g \circ f$ is not surjective.

3. Problem Three: On functions and set operations

Suppose that $f : X \to Y$ is a function and that A and B are subsets of X. Prove the following.

(1) $f(A \cup B) = f(A) \cup f(B)$,
(2) $f(A \cap B) \subseteq f(A) \cap f(B)$,
(3) if f is an injection, then $f(A \cap B) = f(A) \cap f(B)$.

3.1. **Solution:**

(1) Recall that $f(A) = \{x = f(y) \mid y \in A\}$. So we see that

$$f(A \cup B) = \{x = f(y) \mid y \in A \cup B\}$$

and

$$\begin{aligned} f(A) \cup f(B) &= \{x = f(y) \mid y \in A\} \cup \{x = f(y) \mid y \in B\} \\ &= \{x = f(y) \mid y \in A \quad \text{or} \quad y \in B\}. \end{aligned}$$

These are now clearly equal.

(2) $y \in f(A)$ means that there exists $x \in A$ such that $f(x) = y$. So, $y \in f(A \cap B)$ means that there is a point $x \in A \cap B$ such that $f(x) = y$. Since $x \in A$, we see that $y \in f(A)$ and since $x \in B$ we see that $y \in f(B)$. So we have proved the statement.

(3) Now take a point $y \in f(A) \cup f(B)$. $y \in f(A)$ means that there is a point $x_1 \in A$ with $f(x_1) = y$. Similarly, there is a point $x_2 \in B$ such that $f(x_2) = y$. Since f is injective, we know that $x_1 = x_2$, and the point lies in both A and B. Thus, $y \in f(A \cap B)$.

4. Problem Four: On algebraic numbers

A real number α is called *algebraic* when it is the root of some polynomial with integral coefficients. What is the cardinality of the set of algebraic integers? (Back up your answer with a proof.) A number which is not algebraic is called *transcendental*. What is the cardinality of the set of transcendental numbers?

This is not part of the assignment. How many transcendental numbers do you know? My guess is that you know two right off the top of your head. (let them come to you...) Try looking for some others. (Ask around, look into the literature, etc.) Think about this in the context of the answers to this problem for some perspective on your experiences with real numbers. I'll give a small prize to any student who finds a transcendental number I haven't heard about, yet.

4.1. **Solution:** The set of all algebraic numbers is countably infinite. It is infinite because it contains all integers. To see that it is countable, we argue as follows: The set of polynomials with integral coefficients is countable, since it may be identified with a subset of the set of functions from $\mathbb{Z}$ to $\mathbb{Z}$. Each polynomial has finitely many roots. Thus, the set of algebraic integers is a countable union of finite (hence countable) sets, and, therefore, countable.

Since the real numbers can be partitioned into the algebraic numbers and the transcendental numbers, the algebraic numbers being countable and the real numbers being uncountable by Cantor's theorem, we deduce that the set of transcendental numbers is uncountable.

My guess is that you are familiar with e and π which are known to be transcendental. (These are theorems of Hermite (1873) and Lindemann (1882), respectively.) So is $\ln(2)$ (a theorem of Gelfond-Schneider covers this, but I suspect this example was known before). The first one was found by Liouville in 1844, but it is long to describe. Maybe you know some more?

5. Problem Five: On Brevity

Prove that there are real numbers that cannot be defined uniquely in a finite number of words.

5.1. **Solution:** The set of real numbers that can be defined by a finite set of words is countable. There are uncountably many real numbers. Therefore, there exist words that cannot be described in finitely many words.

Notice that the set of algebraic numbers is a subset of the numbers that can be defined in finitely many words. (example: "let α be the smallest positive root of the polynomial...") But some transcendental numbers can also be described in finitely many words, like π, which is the ratio of the circumference to the diameter of a circle. So, these numbers are admittedly hard to think about, but in some sense, most real numbers are this bad.