HOMEWORK ASSIGNMENT 4

MATH 251, WILLIAMS COLLEGE, FALL 2006

Abstract. These are the instructor's solutions.

1. A theorem on sets

Prove that for any sets $A, B_1, \dots, B_n$,

$$A \cap \left(\bigcup_{i=1}^{n} B_i \right) = \bigcup_{i=1}^{n} (A \cap B_i). \tag{1}$$

1.1. **Solution.** We proceed by induction on the number n of sets labeled B_i.
Case One: ($n = 1$) In this case, there is nothing to show, as the two sides of (1) are clearly the same.
Case Two: ($n = 2$) This case is just one of DeMorgan's laws, which we discussed earlier. We must show that

$$A \cap (B_1 \cup B_2) = (A \cap B_1) \cup (A \cap B_2).$$

A point a lies in the left-hand set when $a \in A$ and also $a \in B_1$ or $a \in B_2$. Thus, there are two cases. Either $a \in A$ and $a \in B_1$ or $a \in A$ and $a \in B_2$. This is equivalent to $a \in (A \cap B_1) \cup (A \cap B_2)$. So case two is proved.
Inductive step: Suppose that (1) holds for $n = k$ sets. We are to show that it also holds for $n = k+1$ sets. Using first case two and then the inductive hypothesis, we see that the left hand side of (1) is equal to

$$\begin{aligned}
A \cap \left(\bigcup_{i=1}^{k+1} B_i \right) &= A \cap \left((\bigcup_{i=1}^{k} B_i) \cup B_{k+1} \right) \\
&= A \cap \left(\bigcup_{i=1}^{k} B_i \right) \cup (A \cap B_{k+1}) \\
&= \bigcup_{i=1}^{k} (A \cap B_i) \cup (A \cap B_{k+1}) \\
&= \bigcup_{i=1}^{k+1} (A \cap B_i)
\end{aligned}$$

So, by the principle of induction, we are done.

2. A geometric theorem

Prove that for every integer $n \geq 2$ the number of lines obtained by joining n distinct points in the plane, no three of which are collinear, is $\frac{1}{2}n(n-1)$.

2.1. solution. We proceed by induction.
base case: ($n = 1$) In this case, there is only one point, so there are no lines to draw. The equality is then certainly true.
Inductive step: Suppose that for any configuration of k points, no three of which are collinear, the number of lines that can be drawn joining them is equal to $k(k-1)/2$.

We are to show that for any configuration of $k+1$ points, no three of which are collinear, the number of lines that can be drawn joining them is $(k+1)k/2$.

Suppose we have such a configuration of points $p_1, \ldots, p_{k+1}$. Then the configuration $p_1, \ldots, p_k$ satisfies the inductive hypothesis. Therefore, the number of lines that can be drawn joining all of those points is $k(k-1)/2$. To this, we must add all of the lines which join the remaining point p_{k+1} to the others. There are exactly k of these, since k cannot lie on any of the previous lines. Hence, there are $k(k-1)/2 + k = (k+1)k/2$ lines.

So by the principle of mathematical induction, we are done.

3. A new form of induction

Use either form of induction to prove that the following form of induction is also valid:

> Suppose that $P(n)$ is a statement about the natural number n such that
> (1) $P(1)$ is true,
> (2) for any $k \geq 1$, $P(k)$ true implies $P(2k)$ is also true, and
> (3) for any $k \geq 2$, $P(k)$ true implies that $P(k-1)$ is also true.
> Then $P(n)$ is true for all n.

3.1. Solution. We suppose that the statements itemized above are true, and that the principle of weak induction is true. We use weak induction to prove the statement "$P(m)$ is true for all natural numbers $m \geq 1$."
Base Case: $m = 1$. This is true by the first of the three itemized statements.
Inductive step. Suppose that $P(k)$ is known to be true. We are to show that $P(k+1)$ is also known to be true.

We examine some cases: First, note that if $k = 1$, then $2k = 2 = k+1$, so $P(k+1)$ is true by the second itemized statement.

Next, suppose that $k > 1$. Then by the second itemized statement, $P(2k)$ is also true. We now use the third itemized statement $2k - k - 1 = k - 1$ times to show that $P(2k-1), P(2k-2), \ldots, P(2k-(k-1)) = P(k+1)$ are all true. This last one is the one we need to complete the inductive step.

So, by the principle of mathematical induction, $P(m)$ is true for all $m \geq 1$.

4. The arithmetic-geometric mean inequality

Use the last problem to prove the *arithmetic-geometric mean inequality*: For any $n \geq 1$ and any n nonnegative real numbers $a_1, \ldots, a_n$, we have that

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n}\,. \tag{2}$$

4.1. **Solution.** We use the form of induction from the last problem.
Base case: ($n = 1$). If there is only one number, a_1, then both sides of equation (2) are equal to a_1.

Base case: ($n = 2$). Suppose that a_1, a_2 are nonnegative real numbers. Then we see that

$$0 \leq (\sqrt{a_1} + \sqrt{a_2})^2 = a_1 + a_2 + 2\sqrt{a_1 a_2}\,.$$

With a small bit of rearrangement, we deduce that $\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2}$. Thus, case 2 is true.

First inductive step: Suppose that equation (2) holds for any collection of $n = k$ nonnegative real numbers. We are to show that it holds for any collection of $n = 2k$ nonnegative real numbers.

Let $a_1, a_2, \ldots, a_{2k}$ be such a collection. Then, we see that by the inductive hypothesis and the case $n = 2$ that

$$\begin{aligned}
\frac{a_1 + \ldots + a_{2k}}{2k} &= \frac{1}{2}\left(\frac{a_1 + \ldots + a_k}{k} + \frac{a_{k+1} + \ldots + a_{2k}}{k}\right) \\
&\geq \frac{1}{2}\left(\sqrt[k]{a_1 a_2 \cdots a_k} + \sqrt[k]{a_1 a_2 \cdots a_k}\right) \\
&\geq \sqrt{\sqrt[k]{a_1 a_2 \cdots a_k}\,\sqrt[k]{a_{k+1} a_{k+2} \cdots a_{2k}}} \\
&= \sqrt[2k]{a_1 a_2 \cdots a_{2k}}
\end{aligned}$$

Therefore, equation (2) holds for $n = 2k$ also.

second inductive step: Now we assume that equation (2) holds for $n = k$ and we must show that it also holds for $n = k-1$. Let $a_1, \ldots, a_{k-1}$ be a collection of nonnegative real numbers. We consider the nonnegative real number $b = \frac{a_1 + \cdots + a_{k-1}}{k-1}$. Using the inductive hypothesis, we reason that

$$\begin{aligned}
b = \frac{a_1 + \cdots + a_{k-1}}{k-1} &= \frac{\frac{k}{k-1}(a_1 + \cdots + a_{k-1})}{k} \\
&= \frac{a_1 + \cdots + a_{k-1} + b}{k} \geq \sqrt[k]{a_1 a_2 \cdots b}\,.
\end{aligned}$$

We take the kth power of this equation to see that $b^k \geq a_1 a_2 \cdots b$, so that $b^{k-1} \geq a_1 a_2 \cdots a_{k-1}$. Taking the $(k-1)$st root, we obtain the required inequality:

$$\frac{a_1 + \cdots + a_{k-1}}{k-1} = b \geq \sqrt[k-1]{a_1 a_2 \cdots a_{k-1}}\,.$$

So, by problem 3, the statement is true for all n.