HOMEWORK ASSIGNMENT 5

MATH 251, WILLIAMS COLLEGE, FALL 2006

Abstract. These are the instructor's solutions.

1. Big Brother

The social security number of a person is a sequence of nine digits that are not necessarily distinct. If X is the set of all social security numbers, find the number of elements of X.

1.1. **Solution.**

We have a sequence of 9 choices to make, each from a set of 10 elements, so by the rule of sequential counting, there are 10^9 possibilities.

2. Card Sharks

Find the number of ways of picking the following subsets out of a standard deck of cards:

(1) A king and a queen,
(2) a king or a queen,
(3) a king and a red card,
(4) a king or a red card.

2.1. **Solution.**

There are $4 \cdot 4 = 16$ ways to choose a king and a queen, and $4 + 4 = 8$ ways to pick a king or a queen.

To pick a king and a red card, one can pick one of two black kings and then one of the 26 red cards, or one can pick one of two red kings and then one of the remaining 25 red cards. Thus there are $2 \cdot 26 + 2 \cdot 25 = 102$ different possibilities.

To pick a king or a red card, there are three possible ways. One can pick one of two red kings, one of two black kings, or one of the 24 red cards which are not kings. Thus there are $2 + 2 + 24 = 28$ possibilities.

3. Memory Hungry

A sequence of digits which are either 0 or 1 is called a binary number. A binary number with eight digits is called a *byte*.

(1) Find the number of bytes.
(2) Find the number of bytes that begin with 10 and end with 01.
(3) Find the number of bytes that begin with 10 but do not end with 01.
(4) Find the number of bytes that begin with 10 or end with 01.

3.1. **Solution.** By the rule of sequential counting in each case, there are $2^8 = 256$ total bytes, $2^4 = 16$ bytes that start with 10 and end with 01, and $2^4 \cdot 3 = 48$ bytes that start with 10 but don't end with 01, since this leaves 3 possibilities for the end: 00, 11 and 10.

Using the rule of disjunctive counting, we see that there are $48+48+16 = 112$ bytes for the last case: 48 which start with 10 but don't end with 01, 48 which end with 01 but don't start with 10 and 16 which start with 10 and end with 01.

4. Are we there yet?

There are three bridges connecting two towns, A and B. Between towns B and C there are four bridges. A salesperson has to travel from A to C via B. Find

(1) the number of possible choices for bridges from A to C,
(2) the number of choices for round-trip travel from A to C,
(3) the number of choices for round-trip travel from A to C if no bridge is repeated.

4.1. **Solution.** We use the rule for sequential counting in each case. There are $3 \cdot 4 = 12$ routes from A to C, $3 \cdot 4 \cdot 4 \cdot 3 = 144$ round-trips, and $3 \cdot 4 \cdot 3 \cdot 2 = 72$ round-trips without repeat bridges.

5. On permutations

(1) Show that $P(n, r+1) = (n-r) \cdot P(n, r)$ and use this result to find the value of n if $P(n, 9) = 15P(n, 8)$.
(2) Compute $P(17; 4, 3, 2)$,
(3) Compute $P(17; 2, 2, 2)$

5.1. **Solution.**

(1) $P(n, r+1) = \dfrac{n!}{(n-(r+1))!} = (n-r)\dfrac{n!}{(n-r)!} = (n-r) \cdot P(n, r)$. So if $P(n, 9) = 15P(n, 8)$, we see that $n - 8 = 15$ and $n = 23$.
(2) $P(17; 4, 3, 2) = \dfrac{17!}{4! \cdot 3! \cdot 2! \cdot 8!} = 30,630,600$.
(3) $P(17; 2, 2, 2) = \dfrac{17!}{2! \cdot 2! \cdot 2! \cdot 11!} = 1,113,840$.

6. On Combinations

Compute the following.

1. $C(9,4)$,
2. $C(10,7)$,
3. $C(8,4)$.

6.1. Solution.

1. $C(9,4) = \dfrac{9!}{4! \cdot 5!} = 126$,
2. $C(10,7) = \dfrac{10!}{7! \cdot 3!} = 120$,
3. $C(8,4) = \dfrac{8!}{4! \cdot 4!} = 70$.

7. Why not *West Alabama*?

Find the number of ways to rearrange the letters of $MISSISSIPPI$

1. without restriction,
2. so that all four S's stay together,

7.1. Solution.

1. This is a generalized permutation $P(11;4,4,2,1) = \dfrac{11!}{4! \cdot 4! \cdot 2! \cdot 1!} = 34,560$.
2. Treat the four S's as a unit. Then we are looking at the generalized permutation $P(8;4,2,1,1) = \dfrac{8!}{4! \cdot 2! \cdot 1! \cdot 1!} = 840$.

8. Thanksgiving is coming!

Find the number of ways of seating 14 people such that 8 of them are around a round table and the rest are on a bench.

8.1. **Solution.** View this as a sequence of choices. First, choose a subset of eight people to sit at the table. Then, choose a circular permutation of these eight people to seat them at the table. Finally, choose an ordinary permutation of the remaining six people to sit them on the bench. We obtain

$$C(14,8) \cdot \frac{8!}{8} \cdot 6! = 10,897,286,400$$

arrangements. If you decide the bench has a symmetry and you don't care about strict order but only who sits next to who on the bench, you get half as many arrangements.

9. I hope it's not for dodgeball

A discrete math class consists of 10 math majors and 12 computer science majors. A team of 12 has to be selected from this class. Find the number of ways of selecting a team if

(1) the team has 6 from each discipline,
(2) the team has a majority of computer science majors.

9.1. **Solution.** The first question is a sequential choice. First choose six computer scientists and then choose six mathematicians. There are

$$C(12,6)C(10,6) = 194,040$$

ways to do this.

The second part requires some disjunctive counting first. You must decide on how many computer scientists to include: this is a number between 7 and 12. Thus, in total, there are

$$\begin{aligned} C(12,12)\cdot C(10,0) &+ C(12,11)\cdot C(10,1) + C(12,10)\cdot C(10,2) + \\ &+ C(12,9)\cdot C(10,3) + C(12,8)\cdot C(10,4) + C(12,7)\cdot C(10,5) \\ &= 333,025 \end{aligned}$$

possible teams.

10. This guy again?

Use a combinatorial argument to prove *Newton's identity*:

$$C(n,r)\cdot C(r,k) = C(n,k)\cdot C(n-k,r-k).$$

10.1. **Solution.** The left hand side of the equation is the number of ways to make the following choice: pick r elements out of an n element set, then pick k of those elements to be "extra special". This is clearly the same as picking our k "extra special" elements from the n elements first, then picking $r-k$ elements out of the remaining $n-k$ elements to fill out the set. This can be done in the number of ways given on the right hand side of the equation. The result follows.

11. Something familiar

Prove that

$$C(n,0) + C(n,1) + \cdots + C(n,n-1) + C(n,n) = 2^n.$$

11.1. **Solution.** This is an exercise in disjunctive counting. 2^n is the number of possible subsets of a set of size n. We may choose sets in the following exclusive collection of ways: for each integer k, $0 \le k \le n$, we could pick a subset of size k. There are $C(n,k)$ ways to do this for each k. So by the rule on disjunctive counting, we are done.

Another argument: By the binomial theorem, we see that

$$\begin{aligned} 2^n =&(1+1)^n = \sum_{k=1}^{n} C(n,k)(1)^k(1)^{n-k} \\ & C(n,0) + C(n,1) + \cdots + C(n,n-1) + C(n,n). \end{aligned}$$

12. Something new

Use a combinatorial argument to prove the following identity:

$$[C(n,0)]^2 + [C(n,1)]^2 + \cdots + [C(n,n-1)]^2 + [C(n,n)]^2 = C(2n,n).$$

12.1. **Solution.** Let X be a set with $2n$ elements partitioned into two subsets, Y and Z, each with n elements. We may count the ways to choose a subset of size n out of X by the disjunctive counting rule as follows: For each integer k, one picks k elements of Y to belong to the new set and k elements of Z *not* to belong to the new set. Thus, for each k, we have $C(n,k)C(n,k)$ possibilities. We must add these over k from 0 to n, which gives the result.

13. One Mississippi, Two Mississippi...

Find the number of ways to rearrange the letters of $MISSISSIPPI$ so that no two S's are adjacent.

13.1. **Solution.** View this as a sequence of choices. First, choose a permutation of the non-S letters. This choice can clearly be made in $P(7;4,2,1)$ ways. Then pick an allocation of the 4 S's into the 8 places around these 7 letters. This can be done in $C(8,4)$ ways. Hence the answer is

$$P(7;4,2,1) \cdot C(8,4) = 7,350$$

different ways.

To see this process, imagine that the permutation of the 7 letters has been chosen. We don't care what one it is, so we'll write a generic string: $abcdefg$ of seven letters. There are now eight places in which we could put an S, and at most one S can go in each place. The locations are: before a, after g, and the six places between adjacent letters.