MATH 251 HOMEWORK 6

FALL 2006, WILLIAMS COLLEGE

Abstract. These are the instructors solutions.

1. Collections

Find the number of r-collections that can be formed using the elements of the set $X = \{A, B, C, D, E, F, G\}$ if

- $r = 4$ and the elements are distinct,
- $r = 4$,
- $r = 9$.

1.1. **Solution.** The first choice can be made $C(7, 4) = 35$ ways. If we allow choices to be repeated, we have $C(10, 6) = 210$ possibilities for $r = 4$ and $C(15, 6) = 5,005$ ways for $r = 9$.

2. Equations, part one

Find the number of distinct solutions in nonnegative integers of the equation $a + b + c + d + e = 24$.

2.1. **Solution.** Each solution corresponds to an allocation of 24 things into our 5 labeled locations, a, b, c, d, e. This can be done in $C(24+5-1, 5-1) = C(28, 4) = 20,457$ ways.

3. Don't multiply

Find the number of terms in the multinomial expansion of $(a + b + c + d + e)^{24}$.

3.1. **Solution.** Each term is of the form $Ca^{x_1}b^{x_2}c^{x_3}d^{x_4}e^{x_5}$, where $x_1 + x_2 + x_3 + x_4 + x_5 = 24$. So the number of terms is equal to the number of solutions to this equation. By the last problem, we see there are $20,457$ terms.

4. An Identity

Establish the following identity:

$$C(n,n)+C(n+1,n)+C(n+2,n)+\cdots+C(n+r,n)=C(n+r+1,n+1)$$

4.1. **Solution.** The right-hand side represents the number of distinct solutions in nonnegative integers of the inequality

$$x_1+x_2+\cdots+x_{n+1}\le r. \tag{1}$$

We may divide these solutions into classes depending on the value of x_{n+1}. If $x_{n+1}=r$, there are as many solutions as there are to the inequality $x_1+\cdots+x_n\le 0$. There are $C(n,n)$ of such. In fact, if we let $x_{n+1}=k$, then there are as many solutions to inequality (1) as there are to

$$x_1+\cdots+x_n\le r-k.$$

There are $C(r-k+n,n)$ such solutions. So, by the rule of disjunctive counting, we have

$$\begin{aligned}C(n+r+1,n+1)&=\sum_{k=0}^{r}C(r-k+n,n)\\&=C(n+r,n)+C(n+r-1,n)+\cdots+C(n+1,n)+C(n,n).\end{aligned}$$

This is the same as the sum on the left-hand side of the problem, written in the opposite order.

5. An allocation problem

Find the number of ways of allocating r identical objects to n distinct locations such that the location i gets at least p_i objects, where $i=1,2,\ldots,n$.

5.1. **Solution.** We have to put p_i objects in location i. We then have

$$k=r-p_1-p_2-\cdots-p_n$$

objects left. We are free to allocate these however we want. Thus we are counting the number of k-collections of a set of size n. So the answer is

$$C(k+n-1,n-1)=C(r+n-1-p_1-\cdots-p_n,n-1).$$

6. Equations, part two

Find the number of solutions in nonnegative integers of the strict inequality $a+b+c+d+e<11$.

6.1. **Solution.** We introduce a slack variable f and note that solutions to the inequality correspond to solutions in nonnegative integers of the equation $a+b+c+d+e+f=10$. There are $C(15,5)=3,003$ such solutions.

7. Equations, part three

Find the number of solutions in integers of the linear equation $p + q + r = 25$ where p is at least 2 and at most 4, q is at least 3 and at most 6, and r is at least 4 and at most 8.

7.1. **Solution.** The conditions are impossible to satisfy. There are no solutions. The problem requires $p + q + r \leq 4 + 6 + 8 = 18 < 25$.

8. The first year student mixer

Prove that in any group of 10 people either there is a subgroup of 3 strangers or a subgroup of 4 people known to one another.

8.1. **Solution.** We use the generalized pigeonhole principle. Fix some person x, and consider the rest of the people as a set (with 9 elements). We divide the remaining people into the classes of people who know x or do not. By the generalized pigeonhole principle, either there are at least 4 people who do not know x or there are at least 6 people who do know x. We consider these two cases separately.

Case One: Suppose that there is a group of 4 people who don't know x. If there is a pair of people u, v in this group that don't know each other then the set $\{x, u, v\}$ forms a three element set of strangers and we are done. So suppose that every pair of people in the group knows each other. Since there are 4 of them, we are done.

Case Two: Suppose that there is a set Y consisting of 6 people who know x. **Claim:** either Y has a subset of three mutual strangers, in which case, we are done, or a subset of three mutual friends. If there are three mutual friends in this set, we get the desired set of 4 mutual friends by adding x to the group.

Thus, given the claim, in all possible cases we have reached one of the two conclusions.

So it only remains to prove the claim that In any set of 6 people either there is a group of three strangers, or a group of three mutual friends. Select a person y from the group of 6 people. Since there are 5 other people either there are three people who know y or there are three people who don't know y.

Suppose that there are three people who know y. If this is a group of strangers, we are done. If they are not mutual strangers, then there is a pair of them, say u and v who are friends and then $\{y, u, v\}$ is a set of three mutual friends. Either way the claim is valid.

Now suppose that there are three people who do not know y. If these three form a set of mutual friends we are done, so suppose that they don't form a set of mutual friends. Then there is a pair u, v of people who are strangers, and thus the set $\{u, v, y\}$ forms a set of three mutual strangers. Again, either way the claim is valid.

This proves the claim, and hence the problem is completed.

9. Permutations again

Find the number of permutations of the digits 1, 2, ..., 9 such that

- The blocks 12, 34 and 567 do not appear,
- The blocks 12, 23 and 415 do not appear.

9.1. **Solution.** In each case, we instead compute the number of ways in which the possibilities *can* occur, and then subtract this from 9!, the total number of permutations. The number of ways in which the blocks can appear can be computed using the inclusion-exclusion principle.

(1) The block 12 can occur in 8! ways, since this involves treating 12 as a single symbol–reducing the number of objects to be permuted to 8. Similarly, 34 can occur in 8! ways, and 567 can occur in 7! ways. We then look for double counting: 12 and 34 occur together in 7! ways, the blocks 12 and 567 occur together in 6! ways, and the blocks 34 and 567 occur together in 6! ways. Finally, we account for when all three blocks appear together, this can happen in 5! ways.

Thus, the number of ways these blocks *can* appear is

$$8! + 8! + 7! - 7! - 6! - 6! + 5! = 2(8! - 6!) + 5! = 79,320,$$

and hence the number of permutations in which none of these blocks may appear is

$$9! - 79,320 = 283,560.$$

(2) The block 12 can occur in 8! ways. Similarly, 23 can occur in 8! ways, and 415 can occur in 7! ways. We then look for double counting: 12 and 23 occur together in 7! ways, the blocks 12 and 415 cannot occur together, and the blocks 23 and 415 occur together in 6! ways. Finally, we note that it is impossible for all three blocks to occur together.

Thus, the number of ways these blocks *can* appear is

$$8! + 8! + 7! - 7! - 0 - 6! + 0 = (2 \cdot 8 \cdot 7 - 1)6! = 79,920,$$

and hence the number of permutations in which none of these blocks may appear is

$$9! - 79,920 = 282,960.$$