MATH 251 HOMEWORK 7 SOLUTIONS

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Abstract. These are the instructor's solutions.

1. The Towers of Hanoi

Three vertical cylindrical poles of equal radius and height are place along a line on top of a table and n circular disks of decreasing radius, each with a hole at its center, are attached to the first pole such that the largest one is at the bottom, the next largest is just above that, and so on, until the smallest is on the top of the stack. The distance between the feet of any two poles is not less than the diameter of the largest disk. A legal move is defined as a transfer of the top disk from any one of the poles to another pole as long as no disk is placed upon a smaller disk.

Find and describe an algorithm for moving all of the disks from the first pole to one of the other two poles (a single new stack). Let $f(n)$ be the number of moves required to transfer all the disks as required. Obtain a recurrence relation for $f(n)$ and solve it.

Note: This is a simple way that recurrence relations get used. An algorithm often has recursive parts (as should yours here), so to describe its complexity one can use recurrence relations.

1.1. **Solution.**

Suppose there is only one disk. Then we may simply move it to one of the other poles.

Now suppose there are two disks. Then we first move the top disk to one of the other poles, then move the larger disk to the third pole, and finally we move the small disk to place it on top of the larger disk.

Recursive part: Suppose we have a procedure P for moving n disks. We then may move $n+1$ disks as follows: First, apply P to move the top n disks to some other pole. Then move the largest disk to the third pole. Then reapply P to move the top n disks on top of the largest disk.

If we let $f(n)$ denote the number of moves required to implement this algorithm for n disks, we see that f satisfies the recurrence relation

$$f(n) = 2f(n-1) + 1, \qquad f(0) = 0,\ f(1) = 1.$$

Note that we don't need both initial conditions, we just put them there for clarity.

To solve this, let $g(x)$ be the generating function for $f(n)$. Then

$$\begin{array}{rcccccccc} g(x) & = & f(0) & + & f(1)x & + & f(2)x^2 & + & \dots \\ 2xg(x) & = & & 2f(0)x & +2f(1)x^2 & + & \dots & & \\ \dfrac{1}{1-x} & = & 1 & + & x & + & x^2 & + & \dots \end{array}$$

So, subtracting the second and third equation from the first, we find that

$$g(x)(1-2x) - \frac{1}{1-x} = f(0) - 1 = -1$$

and hence we may solve for the generating function using a partial fraction decomposition as follows

$$\begin{aligned} g(x) =& \frac{-1}{1-2x} + \frac{1}{(1-x)(1-2x)} \\ =& \frac{-1}{1-2x} + \frac{-1}{1-x} + \frac{2}{1-2x} \\ =& \frac{1}{1-2x} - \frac{1}{1-x}. \end{aligned}$$

Since we recognize these as generating functions for simple sequences we can write

$$f(n) = 2^n - 1.$$

It is a simple check to show this solves the recurrence relation.

2. Recurrence Relations

Solve the recurrence relation

$$\begin{cases} a_n & = a_{n-2} + 4n \\ a_0 & = 3 \\ a_1 & = 2 \end{cases}$$

by using a generating function.

2.1. **Solution.**

Let $f(x)$ be the generating function for the sequence a_n. Then we see that

$$\begin{array}{ccccccccccccc} f(x) & = & a_0 & + & a_1 x & + & a_2 x^2 & + & \dots & + & a_n x^n & + & \dots \\ x^2 f(x) & = & & & & & a_0 x^2 & + & \dots & + & a_{n-2} x^n & + & \dots \\ \frac{4x}{(1-x)^2} & = & & & 4 \cdot 1x & + & 4 \cdot 2x^2 & + & \dots & + & 4 \cdot n a_n x^n & + & \dots \end{array}$$

Subtracting the second and third equation from the first, we see that

$$f(x)(1-x^2) - \frac{4x}{(1-x)^2} = a_0 + (a_1 - 4)x = 3 - 2x.$$

We apply the method of partial fractions to this [work compressed out–though I did each term separately] to see that

$$\begin{aligned} f(x) &= \frac{3-2x}{(1+x)(1-x)} + \frac{4x}{(1-x)^3(1+x)} \\ &= \frac{5/2}{1+x} + \frac{1/2}{1-x} + \frac{-1/2}{1+x} + \frac{-1/2}{1-x} + \frac{-1}{(1-x)^2} + \frac{2}{(1-x)^3} \\ &= \frac{2}{1+x} + \frac{-1}{(1-x)^2} + \frac{2}{(1-x)^3} \end{aligned}$$

The first term is recognizable from previous work, the second is the generating function for $-(n+1)$ since

$$\frac{1}{(1-x)^2} = \left[\frac{1}{1-x}\right]' = 1 + 2x + 3x^2 + \dots$$

and the last is the generating function for $(n+1)(n+2)$ since

$$\frac{2}{(1-x)^3} = \left[\frac{1}{(1-x)^2}\right]' = 1\cdot 2 + 2\cdot 3x + 3\cdot 4x^2 + \dots$$

Therefore, we conclude that

$$\begin{aligned} a_n &= 2(-1)^n - (n+1) + (n+1)(n+2) \\ &= 2(-1)^n + n^2 + 2n + 1 \\ &= (n+1)^2 + 2(-1)^n. \end{aligned}$$

Again, it is a simple check using a proof by induction that this solves the recurrence relation.

3. Growth of functions

Prove that for any $b > 1$,

$$\log_b(\log_b(n)) \prec \log_b(n) \prec n.$$

3.1. **Solution.** First, recall that $\log_b(n) = \dfrac{\ln(n)}{\ln b}$.

These follow from L'Hôpital's rule and the theorem about limits from class. We see that

$$\lim_{n\to\infty} \frac{\log_b(n)}{n} = \lim_{n\to\infty} \frac{\frac{1}{n\ln b}}{1} = 0,$$

which shows that $\log_b(n) \prec n$. Similarly, we compute that

$$\lim_{n\to\infty} \frac{\log_b(\log_b(n))}{\log_b(n)} = \lim_{n\to\infty} \frac{\ln(\frac{\ln n}{\ln b})}{\ln n} = \lim_{n\to\infty} \frac{\frac{1}{\ln n}\cdot 1/n}{1/n} = 0,$$

which shows that $\log_b(\log_b(n)) \prec \log_b(n)$.

4. Growth of functions again

Show that $f \asymp g$ is an equivalence relation on the set of functions

$$\{f : \mathbb{N} \to \mathbb{R}_{\geq 0}\}.$$

4.1. **Solution.** Certainly for any function f, we have $f = O(f)$. Therefore, $f \asymp f$, so $\asymp$ is reflexive.

Now suppose that $f \asymp g$. By definition, this means that $f = O(g)$ and $g = O(f)$. Clearly, this also means that $g \asymp f$. Therefore $\asymp$ is symmetric.

Finally, we need to show that $\asymp$ is transitive. Let f, g, h be positive functions with $f \asymp g$ and $g \asymp h$. We are to show that $f \asymp h$. By hypothesis, $f = O(g)$, so there are constants n_0 and $C > 0$ such that $f(n) \leq g(n)$ for $n \geq n_0$. Similarly, $g = O(h)$ means that there are constants n_1 and $C' > 0$ such that $g(n) \leq C'h(n)$ for $n \geq n_1$. Together, these mean that $f(n) \leq CC'h(n)$ for $n \geq \max\{n_0, n_1\}$, hence $f = O(h)$. One can prove $h = O(f)$ similarly. Thus $f \asymp h$, and $\asymp$ is transitive.

Since $\asymp$ is reflexive, transitive and symmetric, it is an equivalence relation.