SECOND EXAM SOLUTIONS

MATH 251, WILLIAMS COLLEGE, FALL 2006

Abstract. These are the instructor's solutions. For statements of the problems, see the posted exam. You should take these more as a quick guide on how to do the problems than as a representative complete solution set.

1. Problem One

We must choose a way to divide the 10 ones in 10=1+1+1+1+1+1+1+1+1+1, into five numbers. This involves putting in four "gaps" and choosing their locations. Thus there are $C(10+4,4) = C(14,4) = 91$ such solutions.

If we assume that the solutions must be in positive integers, then none can be zero. We let $a = a'+1, b = b'+1, c = c'+1, d = d'+1, e = e'+1$. Then a', b', c', d', e' are any nonnegative integer but they must satisfy

$$10 = a' + 1 + b' + 1 + c' + 1 + d' + 1 + e' + 1 = a' + b' + c' + d' + e' + 5,$$

or $a' + b' + c' + d' + e' = 5$. This can be solved like the last part of this problem. There are $C(5+4,4) = C(9,4) = 126$ possible solutions.

For this part, we realize that we are in the same situation as the last part, but that a' must take one of the values $0, 2, 4$. (These are the only ways to make a odd and still have all the unknowns positive.) Thus we are counting the solutions to the equations $0+b'+c'+d'+e' = 5$, $2+b'+c'+d'+e' = 5$ and $4+b'+c'+d'+e' = 5$. Therefore, there are $C(8,3) + C(6,3) + C(4,3) = 80$ solutions.

2. Problem Two

First, we choose which set we will use first to make a linear arrangement–there are 2 possibilities. Then we must interlace a pair of n-permutations of sets of size n. There are $n! \cdot n!$ ways to choose such an arrangement. When placing them around a table, we introduce rotational symmetry. In effect, we no longer care which of the $2n$ places is our starting point, so we divide out by this set of symmetries to get $2(n!)^2/(2n) = n!(n-1)!$ ways to arrange the sets in the prescribed fashion.

3. Problem Three

The type of proposition that can be vacuously true is an implication $p \to q$. This happens when the premise of the implication is never true. The concept relies on the fact that the truth value of $p \to q$ will be true if p is false, no matter what what the value of q is. An example of a vacuously true statement is

If my house cat weighs 10,000 pounds, then I have blue hair.

This statement is always true, exactly because I never have to worry about the premise being valid. (I don't even have a cat, let alone such a large one.)

4. Problem Four

The right hand side is the number of ways to choose n objects from a set X of size $m+n$. We partition X into sets A of size m and B of size n. Then there are several mutually exclusive ways to make this choice. For each integer k between 1 and n we shall make a choice of k elements of A to belong to the set and then a choice of k elements of B not to belong to the set. Since B has n elements, this puts $n-k$ elements into our set, making the required total of n. But for each k, this is a sequence of two independent choices. The first can be done in $C(m,k)$ ways and the second in $C(n,k)$ ways. So, combining the rule of sequential counting for each k with the rule of disjunctive counting for the exclusive cases, we get the identity.

5. Problem Five

Suppose that $f : A \to B$ is invertible with inverse $g : B \to A$. By definition, we have that $f \circ g = 1_B$ and $g \circ f = 1_A$.

Let x, y be elements of A such that $f(x) = f(y)$. Then

$$x = g(f(x)) = g(f(y)) = y.$$

Hence, f is injective.

Let $b \in B$. Let $a = g(b)$. Then $f(a) = f(g(b)) = b$, so $b \in \mathrm{range}(f)$. Since b was arbitrary, we see that f is surjective.

Now suppose that f is bijective. We define a function $g : B \to A$ as follows. Since f is surjective, for an element $b \in B$, there exists a point $a \in A$ such that $f(a) = b$. Since f is injective, this element is unique! Therefore, it makes sense to define $g(b)$ as the unique element a such that $f(a) = b$ (that is, this is an unambiguous rule).

We must show that g is the inverse of f. Let $b \in B$. Then $f(g(b)) = f(a) = b$ because $a = g(b)$ is the unique element of A such that $f(a) = b$. Thus, $f \circ g = 1_B$. Now consider a point $a \in A$. Then $g(f(a))$ is the unique point x in A such that $f(x) = f(a)$. Since f is injective, we see that $x = a$. Hence $g \circ f = 1_A$.

Many of you successfully used the following alternate route: Study the inverse relation $f^{-1} \subset B \times A$ and recall the requirement for a relation to be a function: a relation R is a function if and only if every element of the domain appears in exactly one pair in R.

6. Problem Six

We work with the inclusion-exclusion principle. There are $n!$ total permutations. How many fix a set of symbols $i_1, i_2, \ldots, i_k$? Each of these is a permutation of the other $n-k$ elements!

So for each one of the $C(n,1) = n$ elements, there are $(n-1)!$ permutations.

In fact for each of the $C(n,k)$ subsets of k elements, there are $(n-k)!$ permutations which fix exactly those elements.

So by the principle of inclusion-exclusion, we see that there are

$$\begin{aligned} n! - C(n,1)\cdot(n-1)! &+ C(n,2)\cdot(n-2)! - \cdots + (-1)^n C(n,n)\cdot 1! \\ &= n!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right) \end{aligned}$$

derangements of a sequence of length n.

7. Problem Seven

We consider the 30 people as locations to which we should distribute a set of check marks, one mark for each relationship they are in. If there are 104 pairs of acquaintances, there are 208 check marks, since each pair involves 2 people. So we see that there must be at least one person with $\text{floor}(208/30) = 7$ relationships. Here $\text{floor}(x)$ is the floor function.

An alternate argument: If there were only 6 relationships per person, we would only have $6 * 30 = 180$ check marks.

Now suppose that every person has at least seven acquaintances. Then there must be at least $7 * 30 = 210$ check marks, and hence 105 pairs of acquaintances. This is a contradiction, so there must be one person with fewer than 7 acquaintances.