ZEN AND THE ART OF PROOF WRITING

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This is a set of notes on how to start a proof without actually knowing what you are doing. These should be used as a template for getting out of a "please prove this"-induced paralysis.

1. The template and its parts

Here, I make a list of the parts of the template to follow, in order.

(1) State the result you are trying to prove carefully and clearly. It doesn't matter if you label it Theorem, Lemma, Proposition, Claim, Corollary, or whatever else, but it must be clear.
(2) Choose a method of proof. The available styles: direct, by cases, by contradiction, contrapositive, induction (see section 2 for more on this).
(3) Begin your proof by stating your assumptions. This often sets some notation for things you will be talking about.
(4) State what it is you are to prove. Use the notation you just established. This may be complicated by the choice of method above.
(5) Rephrase what you must prove *as an implication* $p \to q$.
(6) State that you are assuming the premise(s) p of the implication.
(7) Working step by step from those premises, using known definitions and theorems, work towards the truth of the conclusion of the implication. It helps to use sentences like:

    Since x is true, we see that...

    or

    By the theorem on y, we know...
(8) Be sure to allude to the punch-line conclusion q at the end of your argument. ("Therefore...")
(9) State clearly that you are done. Many people like to say it clearly and then put a little symbol to alert the reader that things should be finished.

2. On induction

Recall that proofs by induction have their own special structure. It is important to adhere to that, and to do so clearly. Say that you will use induction. Prove the relevant base case(s). Say the inductive step is coming up, and clearly state your inductive hypothesis. State when you are done.

In each of the steps, it may help to use the template above.

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3. An example with the template highlighted.

I've taken a homework exercise and written a proof with the parts of the template in bold type. (They are in the order above.) Until you are more comfortable writing proofs, it may be helpful to use the same type of language as I use here.

Theorem 3.1. *The composition of two surjective functions is also surjective.*

Proof: **Assume that** the functions $f : A \to B$ and $g : B \to C$ are surjective. **We are to show that** $g \circ f : A \to C$ is surjective. **That is, we must show if** c is in C, **then** there exists an a in A such that $g \circ f(a) = c$. So, **suppose that** $c \in C$.

Since g **is surjective**, there is a point b in B such that $g(b) = c$. **Since** f **is surjective**, there is a point a in A such that $f(a) = b$. Now note that $g \circ f(a) = g(f(a)) = g(b) = c$. **Therefore**, there exists a point a in A such that $g \circ f(a) = c$. **We conclude that** $g \circ f$ is surjective. ■

4. Other advice

This section just collects some random advice that doesn't fit into the program above.

- If you are stuck, try writing down the technical definitions of the terms involved on a separate piece of paper.
- If you are stuck, try to draw a representative picture or diagram about the problem.
- If you are stuck, it can help to reduce the conditions of the premise and/or conclusion to "atomic" parts. That is, break things up into small manageable pieces.
- If you are really stuck, try a different method of proof.
- If you are stuck, try to attack a special case first. That is, find something a bit smaller or simpler to work on first. Maybe you can solve that one and gain insight into the larger problem.
- Have you used all of your hypotheses? Have you accidentally used any others?
- If you are proving an equality or an inequality, you may not mix sides of your equation willy-nilly. It is best to start with one side and manipulate it until it looks like the other side.

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