Math 101, Fall 2005, Homework #10

- **Apply the evaluation theorem to evaluate the integrals in Problems 1-30.**

5.5.1 $\int_0^1 (3x^2 + 2\sqrt{x} + 3\sqrt[3]{x})\, dx$

Split up the sum and rewrite each integral to make them easier to solve:

$$\begin{aligned}\int_0^1 \left(3x^2 + 2\sqrt{x} + 3\sqrt[3]{x}\right) dx &= \int_0^1 x^2\ dx + \int_0^1 2x^{1/2}\ dx + \int_0^1 3x^{1/3}\ dx \\ &= 3\int_0^1 x^2\ dx + 2\int_0^1 x^{1/2}\ dx + 3\int_0^1 x^{1/3}\ dx \\ &= 3\left[\frac{x^3}{3}\right]_0^1 + 2\left[\frac{x^{3/2}}{3/2}\right]_0^1 + 3\left[\frac{x^{4/3}}{4/3}\right]_0^1 \\ &= 3\left[\frac{1}{3} - 0\right] + 2\left[\frac{2}{3} - 0\right] + 3\left[\frac{3}{4} - 0\right] \\ &= 1 + \frac{4}{3} + \frac{9}{4} = \frac{55}{12}\end{aligned}$$

5.5.3 $\int_0^1 x^3(1+x)^2 dx$

Multiply out the integrand to make this the integral of a polynomial, then split up the sum:

$$\begin{aligned}\int_0^1 x^3(1+x)^2 dx &= \int_0^1 x^3\left(1 + 2x + x^2\right)\ dx \\ &= \int_0^1 \left(x^3 + 2x^4 + x^5\right)\ dx \\ &= \int_0^1 x^3\ dx + 2\int_0^1 x^4\ dx + \int_0^1 x^5\ dx \\ &= \left[\frac{x^4}{4}\right]_0^1 + 2\left[\frac{x^5}{5}\right]_0^1 + \left[\frac{x^6}{6}\right]_0^1 \\ &= \frac{1}{4} + \frac{2}{5} + \frac{1}{6} = \frac{49}{60}\end{aligned}$$

5.5.5 $\int_0^1 (x^4 - x^3)\, dx$

Just split up the integrand:

$$\begin{aligned}\int_0^1 \left(x^4 - x^3\right) dx &= \int_0^1 x^4\ dx - \int_0^1 x^3\ dx \\ &= \left[\frac{x^5}{5}\right]_0^1 - \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}\end{aligned}$$

5.5.7 $\int_{-1}^{0}(x+1)^3 dx$

Let $u(x) = x+1$, so $du = dx$, then substitute into the integral. Remember to adjust the limits of integration to reflect the variable u.

$$\int_{-1}^{0}(x+1)^3 dx = \int_{u(-1)}^{u(0)} u^3\ dx = \int_{0}^{1} u^3\ dx$$
$$= \left.\frac{u^4}{4}\right|_0^1 = \frac{1}{4}$$

5.5.9 $\int_0^4 \sqrt{x}\ dx$

Rewrite the root and use power rule for integration:

$$\int_0^4 \sqrt{x}\ dx = \int_0^4 x^{1/2}\ dx = \left.\frac{x^{3/2}}{3/2}\right|_0^4$$
$$= \frac{4^{3/2}}{3/2} = \frac{2\left(\sqrt{4}\right)^3}{3} = \frac{16}{3}$$

5.5.11 $\int_{-1}^{2}(3x^2+2x+4)\,dx$

Again, just split up the sum:

$$\int_{-1}^{2}\left(3x^2+2x+4\right) dx = \int_{-1}^{2} 3x^2\ dx + \int_{-1}^{2} 2x\ dx + \int_{-1}^{2} 4\ dx$$
$$= \left[x^3\right]_{-1}^{3} + \left[x^2\right]_{-1}^{3} + [4x]_{-1}^{3} = 8+1+4-1+8+4$$
$$= 24$$

5.5.26 $\int_5^{10} \frac{1}{x}\ dx$

Use the fact that the integral of $\frac{1}{x}$ is $\ln x$:

$$\int_5^{10} \frac{1}{x}\ dx = \ln x|_5^{10}$$
$$= \ln 10 - \ln 5 = \ln 2$$

5.5.28 $\int_0^{\pi/2} \cos 2x\ dx$

Let $u(x) = 2x$, so $du = 2dx$ and $dx = \frac{1}{2}du$. Then we have the substitution:

$$\int_0^{\pi/2} \cos 2x\ dx = \int_0^{\pi} \cos u \cdot \frac{1}{2}du$$
$$= \frac{1}{2}\sin u\Big|_0^{\pi} = \frac{1}{2}(0-0) = 0$$

5.5.30 $\int_0^\pi \sin^2 x \cos x \; dx$

There is a small issue with the limits of integration here, so we'll solve the indefinite integral first, and then evaluate using the limits of integration.

Let $u(x) = \sin x$, so $du = cosx \; dx$. Then we have the substitution:

$$\begin{aligned} \int \sin^2 x \cos x \; dx &= \int u^2 \; du \\ &= \frac{u^3}{3} = \frac{1}{3}\sin^3 x \end{aligned}$$

Then the definite integral is

$$\int_0^\pi \sin^2 x \cos x \; dx = \frac{1}{3}\sin^3 x \Big|_0^\pi = 0 - 0 = 0$$

5.5.37 Evaluate the limit $\lim_{n\to\infty} \sum_{i=1}^n \left(\frac{2i}{n} - 1\right)\frac{1}{n}$ **by first recognizing the indicated sum as a Riemann sum associated with a regular partition of** $[0,1]$ **and then evaluating the corresponding integral.**

Since this is a regular partition of $[0,1]$, we have $\Delta x = \frac{1-0}{n} = \frac{1}{n}$. Then, if we let $x_i = a + i \cdot \Delta x = \frac{i}{n}$, this sum becomes

$$\lim_{n\to\infty} \sum_{i=1}^n \left(\frac{2i}{n} - 1\right)\frac{1}{n} = \lim_{n\to\infty} \sum_{i=1}^n (2x_i - 1)\Delta x = \int_0^1 (2x-1) \; dx$$

Split up the integrand to simplify:

$$\begin{aligned} \int_0^1 (2x-1) \; dx &= \int_0^1 2x \; dx - \int_0^1 \; dx \\ &= \left[x^2\right]_0^1 - [x]_0^1 = 1 - 1 = 0 \end{aligned}$$

Therefore, we have

$$\lim_{n\to\infty} \sum_{i=1}^n \left(\frac{2i}{n} - 1\right)\frac{1}{n} = \int_0^1 (2x-1) \; dx = 0$$

- **For problems 3 and 4, find the average value of the given function on the specified interval.**

5.6.3 $h(x) = 3x^2\sqrt{x^3+1}; \quad [0,2]$

From the definition on page 350, the average value on the interval is given by:

$$\bar{y} = \frac{1}{2-0}\int_0^2 h(x) \; dx = \frac{1}{2}\int_0^2 3x^2\sqrt{x^3+1} \; dx$$

Let $u(x) = x^3 + 1$, so $du = 3x^2 dx$. Then we have the substitution:

$$\begin{aligned}\bar{y} &= \frac{1}{2}\int_1^9 \sqrt{u}\ du = \frac{1}{2} \cdot \frac{u^{3/2}}{3/2}\bigg|_1^9 \\ &= \frac{9^{3/2}}{3/2} - \frac{1^{3/2}}{3/2} = \frac{26}{3}\end{aligned}$$

5.6.4 $f(x) = 8x; \quad [0, 4]$

As in the previous problem, the average value on the interval is:

$$\bar{y} = \frac{1}{4-0}\int_0^4 8x\ dx = \frac{1}{4} \cdot 4x^2\bigg|_0^4 = 16$$

5.6.29 Find the total area of the region bounded below by the x-axis and above by the function $f(x)$, where $f(x) = 1 - x^4$ for $x \leq 0$ and $f(x) = 1 - x^3$ for $x \geq 0$.

As in Figure 5.6.8, $f(x) = 1 - x^4$ has an x-intercept at $x = -1$, and $f(x) = 1 - x^3$ has an x-intercept at $x = 1$. Then we can find the area of the defined region as the integral $A = \int_{-1}^{1} f(x)\ dx$. Since f is defined piecewise, we can split this integral up and evaluate on $[-1, 0]$ and $[0, 1]$ separately. Then we have:

$$\begin{aligned}A &= \int_{-1}^{1} f(x)\ dx = \int_{-1}^{0} f(x)\ dx + \int_0^1 f(x)\ dx \\ &= \int_{-1}^{0} \left(1 - x^4\right)\ dx + \int_0^1 \left(1 - x^3\right)\ dx \\ &= \int_{-1}^{0} dx - \int_{-1}^{0} x^4\ dx + \int_0^1 dx - \int_0^1 x^3\ dx \\ &= [x]_{-1}^{0} - \left[\frac{x^5}{5}\right]_{-1}^{0} + [x]_0^1 - \left[\frac{x^4}{4}\right]_0^1 \\ &= 1 - \frac{1}{5} + 1 - \frac{1}{4} \\ &= \frac{31}{20}\end{aligned}$$

- **For Problems 11-45, evaluate the indefinite integrals.**

5.7.11 $\int (x+1)^6 dx$

Let $u(x) = x + 1$, so $du = dx$. Then we have the substitution:

$$\begin{aligned}\int (x+1)^6\ dx &= \int u^6\ dx = \frac{u^7}{7} + b \\ &= \frac{(x+1)^7}{7} + C\end{aligned}$$

5.7.13 $\int (4-3x)^7 dx$

Let $u(x) = 4-3x$, so $du = -3dx$ and $dx = -\frac{1}{3}du$. Then we have the substitution:

$$\int (4-3x)^7 dx = -\frac{1}{3}\int u^7\ du = -\frac{1}{3}\cdot\frac{u^8}{8} + b$$
$$= -\frac{(4-3x)^8}{24} + C$$

5.7.15 $\int \frac{dx}{\sqrt{7x+5}}$

Let $u(x) = 7x+5$, so $du = 7dx$ and $dx = \frac{1}{7}du$. Then we have the substitution:

$$\int \frac{dx}{\sqrt{7x+5}} = \frac{1}{7}\int u^{-1/2}\ du = \frac{u^{1/2}}{7/2} + b$$
$$= \frac{2\sqrt{7x+5}}{7} + C$$

5.7.17 $\int \sin(\pi x + 1)dx$

Let $u(x) = \pi x + 1$, so $du = \pi dx$ and $dx = \frac{1}{\pi}du$. Then we have the substitution:

$$\int \sin(\pi x+1)dx = \frac{1}{\pi}\int \sin u\ du = -\frac{1}{\pi}\cos u + b$$
$$= -\frac{1}{\pi}\cos(\pi x + 1) + C$$

5.7.19 $\int \sec 2\theta \tan 2\theta\ d\theta$

Let $u(\theta) = \sec 2\theta = \frac{1}{\cos 2\theta}$, so

$$du = \frac{2\sin 2\theta}{\cos^2 2\theta}\ d\theta = \frac{2}{\cos 2\theta}\cdot\frac{sin 2\theta}{\cos 2\theta}\ d\theta = 2\sec 2\theta\tan 2\theta\ d\theta$$

and so $\frac{1}{2}du = \sec 2\theta \tan 2\theta\ d\theta$. Then we have the substitution:

$$\int \sec 2\theta\tan 2\theta\ d\theta = \frac{1}{2}\int du = \frac{u}{2} + b = \frac{\sec 2\theta}{2} + C$$

5.7.21 $\int e^{1-2x}dx$

Let $u(x) = 1-2x$, so $du = -2dx$ and $dx = -\frac{1}{2}du$. Then we have the substitution:

$$\int e^{1-2x}dx = -\frac{1}{2}\int e^u\ du = -\frac{1}{2}e^u + b = -\frac{1}{2}e^{1-2x} + C$$

5.7.45 $\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$ [***Suggestion*****: Try** $u = \sqrt{x}$**.**]

Let $u(x) = \sqrt{x}$, so $du = \frac{dx}{2\sqrt{x}}$ and $2du = \frac{dx}{\sqrt{x}}$. Then we have the substitution:

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx = 2\int \cos u\ du = 2\sin u + b = 2\sin\sqrt{x} + C$$

- **For Problems 51-55, evaluate the definite integrals.**

5.7.51 $\int_1^2 \frac{dt}{(t+1)^3}$

Let $u(t) = t+1$, so $du = dt$. Then we have the substitution:

$$\int_1^2 \frac{dt}{(t+1)^3} = \int_2^3 \frac{du}{u^3} = -\frac{1}{2u^2}\bigg|_2^3 = -\frac{1}{18} + \frac{1}{8} = \frac{5}{72}$$

5.7.53 $\int_0^4 x\sqrt{x^2+9}\ dx$

Let $u(x) = x^2+9$, so $du = 2xdx$ and $\frac{1}{2}du = xdx$. Then we have the substitution:

$$\int_0^4 x\sqrt{x^2+9}\ dx = \frac{1}{2}\int_9^2 5\sqrt{u}\ du = \frac{1}{2}\cdot\frac{u^{3/2}}{3/2}\bigg|_9^2 5$$
$$= \frac{25^{3/2}}{3} - \frac{9^{3/2}}{3} = \frac{98}{3}$$

5.7.55 $\int_0^8 t\sqrt{t+1}\ dx$ [***Suggestion*: Try** $u = t+1$.]

Let $u(t) = t+1$, so $du = dt$ and $t = u-1$. Then we have the substitution:

$$\int_0^8 t\sqrt{t+1}\ dx = \int_1^9 (u-1)\sqrt{u}\ du = \int_1^9 \left(u^{3/2} - u^{1/2}\right)\ du$$
$$= \int_1^9 u^{3/2}\ du - \int_1^9 u^{1/2}\ du = \left[\frac{u^{5/2}}{5/2}\right]_1^9 - \left[\frac{u^{3/2}}{3/2}\right]_1^9$$
$$= \frac{9^{5/2}}{5/2} - \frac{2}{5} - \frac{9^{3/2}}{3/2} + \frac{2}{3} = \frac{1192}{15}$$