Math 102: Second Exam Solutions

Problem 1. There are many correct answers, here are a few easy ones.

a) $\sum_{n=1}^{\infty}(-1)^n\sqrt{n}$ b) $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ c) $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$

Problem 2. The comparison test is:

Suppose that $\sum a_n$ and $\sum b_n$ are two series with positive terms and that $a_n \leq b_n$ for all n which are sufficiently large. Then we have

- If $\sum a_n$ diverges, then so does $\sum b_n$;
- If $\sum b_n$ converges, then so does $\sum a_n$.

Problem 3. Convergence/divergence for numerical series.

a) This diverges. The easiest way is to use a limit comparison test comparing with the divergent p-series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$.

b) This converges. I think the easiest way is to use the integral test and compare with the convergent indefinite integral $\int_2^{\infty}\frac{dx}{x\cdot(\ln x)^{3/2}}$. You should check that the integrand here is positive, decreasing, goes to zero as x gets large, and has $f(n) = a_n$.

Problem 4. Power series question #1.

a) This is a simple ratio test example. The radius of convergence is $R = \infty$, so the interval of convergence is the whole line $(-\infty, \infty)$. Alternately, you can recognize this as the derivative of the power series for xe^x.

b) Again the ratio test is useful here. The radius of convergence is then $R = 1/2$. In testing the endpoints, we see that $x = 1/2$ gives us an series which converges by the alternating series test, and at $x = -1/2$, we find a divergent p-series where $p = 1/2$. Thus the interval of convergence is $(-1/2, 1/2]$.

Problem 5. Power series qusetion #2.

a) This was a homework problem and the solution is posted in my solutions to homework #8.

b) This can be done by termwise differentiation. The key is that the function $J(x)$ is equal to $x^2\cdot\frac{d}{dx}\left(\frac{1}{1-x}\right)$. Recalling the series for this simpler example, we get

$$J(x) = x^2\frac{d}{dx}\left(\sum_{n=0}^{\infty}x^n\right) = x^2\left(\sum_{n=0}^{\infty}nx^{n-1}\right) = \sum_{n=0}^{\infty}nx^{n+1}$$

The radius of convergence is 1, which is inherited from the geometric series we started with.

Problem 6. There are several ways to do this problem. One is to use a remainder term from Taylor's theorem, since the Binomial series is a Taylor series. Another would be to set up an integral test looking estimate. The quickest way to an answer is to use the alternating series error estimate. The binomial series eventually alternates for our exponent $\alpha = 1/2$. Thus the error is less than the size of the next ($n = 4$) term. We get

$$|\text{Error}| \leq 2\left|\frac{(\frac{1}{2})(\frac{-1}{2})(\frac{-3}{2})(\frac{-5}{2})}{4!}\cdot\left(\frac{1}{4}\right)^4\right| = 2\frac{5}{2^{15}} \approx 0.00031$$

Problem 7. There are many ways to do this. My favorite is: $\sum\frac{(-1)^n}{\sqrt{n}}$ converges by the alternating series test, but $\sum a_n^2 = \sum\frac{1}{n}$ is the harmonic series, which diverges.