Math 211: First Exam Solutions

Problem 1:

A: Third Order.
B: (i) $\cos(x)y' + \sin(x)y - \tan(x) = 0$ is linear.
C: (ii) $\dfrac{dy}{dt} = 6y^2 - \ln(\cos(y))$ is autonomous.
D: $y = x\ln(x) - x + 10$ is not a solution to the equation $e^{y'} = x^2$.
E: Lots of answers work here. One really simple example is $y' + y = 0$.

Problem 2: The associated homogeneous equation is $y' + \dfrac{2}{x}y = 0$. This is separable, and can be rearranged to read $\dfrac{dy}{y} = -\dfrac{2dx}{x}$. This has solution $y = Cx^{-2}$. We vary the parameter C and look for a solution to the original equation of the form $y = v(x)/x^2$, where $v(x)$ is a function to be determined. Substituting yields the equation for v:

$$\left(\frac{v'}{x^2} + \frac{-2v}{x^3}\right) + \frac{2v}{x^3} = 8x.$$

Equivalently, we get another separable equation $v' = 8x^3$. This has solution $v(x) = 2x^4 + C$. Putting all of this together, we get that our general solution is $y(x) = 2x^2 + C/x^2$.

Problem 3: First we rearrange the equation to read $y' - 6y = -e^x$. Our integrating factor will be a function u which satisfies $u'/u = -6$. Thus, we take $u(x) = e^{-6x}$. Using this, we have

$$(e^{-6x}y)' = e^{-6x}y' - 6e^{-6x}y = e^{-6x}(y' - 6y) = e^{-6x} \cdot (-e^x) = -e^{-5x}.$$

We integrate this to find that $e^{-6x}y = e^{-5x}/5 + C$, which means $y(x) = e^x/5 + Ce^{6x}$. To match the initial condition, we need $C = -1/5$. So our particular solution is

$$y(x) = (e^x - e^{6x})/5.$$

The interval of existence of this solution is the whole line $(-\infty, \infty)$.

Problem 4:

A: In differential form, the equation is $(y^2/x)\ dx + (2y\ln(x) + 1)\ dy = 0$. That is, $P\ dx + Q\ dy = 0$ for $P = y^2/x$ and $Q = 2y\ln(x) + 1$. We then check that $\dfrac{\partial P}{\partial y} = 2y/x$ and $\dfrac{\partial Q}{\partial x} = 2y/x$. Since these are equal, the form is exact.

B: To solve the equation, we first integrate P with respect to x.

$$F(x,y) = \int P\ dx = \int y^2/x\ dx = y^2 \ln|x| + g(y).$$

To find the function $g(y)$, we differentiate the result above with respect to y

$$2y\ln(x) + 1 = Q = \frac{\partial F}{\partial y} = 2y\ln|x| + g'(y).$$

This means that $g'(y) = 1$. So we take $g(y) = y$, and our solutions should be defined implicitly by the level sets

$$F(x,y) = y^2\ln|x| + y = C.$$

Where C is an arbitrary constant.

Problem 5:

A: The proper differential equation is $M'(t) = -M + 3M - 2 = 2(M-1)$.

B: The equilibrium points are solutions to $M - 1 = 0$. That is $M = 1$. It is not hard to see that this is an unstable point. I'll omit the actual picture of the phase line.

If David starts with three dollars, then his cash on hand M will continue to grow as time increases because he is above the unstable equilibrium. It is worth the effort.

Problem 6:

A: The function $f(x,t) = tx^{2/3}$ is continuous in the whole tx-plane. So by the Existence theorem, there is some solution for every possible initial condition.

B: The function f has $\partial f/\partial x = \dfrac{2t}{3x^{1/3}}$. This is not continuous along the line $x = 0$, but is continuous everywhere else. So by the Uniqueness theorem, the only problem spots must occur along the line $x = 0$. It is then easy to check that $x(t) = 0$ is a solution to the equation. Of course, the equation is separable, so it is not hard to see that a general solution is $x(t) = \left(\dfrac{t^2}{6} + C\right)^3$. By picking $C = 0$, we get two solutions to the equation which satisfy the initial condition $x(0) = 0$. They are $x(t) = 0$ and $x(t) = \dfrac{t^6}{216}$. Of course, there are other choices for this last part.

Problem 7: In this problem, we match as follows: equation one corresponds to direction field A, equation two corresponds to direction field C, and equation three corresponds to direction field B.