Homework #10 Solutions

1. Exercises from the Text

1.1. Section 9.5.

Problem 3. We are to use the definition to compute the exponential of the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We need to know some of the powers of A, so we compute them. The square of this matrix is $A^2 = \mathbf{0}$, all of the powers above this vanish, too. This means that

$$e^A = I + A + 0 = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Problem 9. Consider the matrices $A = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$.

(a): We see that $AB = \begin{pmatrix} -4 & 0 \\ 0 & 0 \end{pmatrix}$ but $BA = \begin{pmatrix} 0 & 0 \\ 0 & -4 \end{pmatrix}$.

(b): Note that $A + B = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} = 2\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. So that

$$e^{A+B} = e^{2\cdot\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$$

To evaluate this last exponential, we note that

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hence, if $t = 2$,

$$\begin{aligned} e^{A+B} &= e^{t\cdot\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} \\ &= I + t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 + \frac{t^3}{3!}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^3 + \frac{t^4}{4!}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^4 + \dots \\ &= \begin{pmatrix} 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots & t - \frac{t^3}{3!} + \dots \\ -(t - \frac{t^3}{3!} + \dots) & 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots \end{pmatrix} \\ &= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} = \begin{pmatrix} \cos 2 & -\sin 2 \\ \sin 2 & \cos 2 \end{pmatrix} \end{aligned}$$

(c): Since both $A^2 = \mathbf{0}, B^2 = \mathbf{0}$, it is clear that $e^A = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ and $e^B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, so that

$$e^A \cdot e^B = \begin{pmatrix} -3 & -2 \\ 2 & -3 \end{pmatrix}.$$

This teaches us that the usual rule for exponentials need not apply if the matrices don't commute.

Problem 17. We are given that the matrix

$$A = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & -1 \\ -2 & 4 & -3 \end{pmatrix}$$

has only one eigenvalue λ. We are to find the smallest integer k such that $(A - \lambda I)^k = 0$ and compute e^A.

The characteristic polynomial of A is $p_A(t) = (-1-t)((1-t)(-3-t)+4) = -(1+t)(t^2+2t+1) = -(t+1)^3$, thus the eigenvalue of A is $\lambda = -1$. We then compute that

$$A + I = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 2 & -1 \\ -2 & 4 & -2 \end{pmatrix}, \quad \text{and} \quad (A+I)^2 = \mathbf{0}.$$

Now by Proposition 5.19, we see that

$$e^{tA} = e^{-t}(I + t(A+I)) = \begin{pmatrix} e^{-t} & 0 & 0 \\ -te^{-t} & e^{-t}(1+2t) & -te^{-t} \\ -2te^{-t} & 4te^{-t} & e^{-t}(1-2t) \end{pmatrix}$$

Problem 25. In this problem, we are to consider the matrix

$$A = \begin{pmatrix} -2 & 1 & -1 \\ 1 & -3 & 0 \\ 3 & -5 & 0 \end{pmatrix}.$$

- To find the eigenvalues, we compute the characteristic polynomial. It is $p_A(t) = -1(-5-3(-3-t)) - 0 + (-t)((-2-t)(-3-t)-1) = -(t^3+5t^2+8t+4)$. Let's forget the negative sign, which won't change the roots. Note that $t = -1$ is a root (by inspection). So we divide by $t+1$ to get $p_A(t) = (t+1)(t^2+4t+4) = (t+1)(t+2)^2$. Thus our eigenvalues are $\lambda_1 = -1, \lambda_2 = -2$.

- The algebraic multiplicity of $\lambda_1 = -1$ is one, so the geometric multiplicity must also by one. We find that a corresponding eigenvector is $\mathbf{v}_1 = \begin{pmatrix} 2 & 1 & -1 \end{pmatrix}^T$.
- The algebraic multiplicity of $\lambda_2 = -2$ is two, but it has only one corresponding eigenvector (up to scaling), $\mathbf{v}_2 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$. The nullspace of $(A+2I)^2$ has dimension two, and we only need to pick one generalized eigenvector to span this space. The vector $\mathbf{v}_3 = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}^T$.
- To see that the three vectors we have chosen are linearly independent, we compute that $\det(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = 1 \neq 0$.
- We get a fundamental set of solutions to $y' = Ay$ as

$$\mathbf{x}_1(t) = e^{-t}\mathbf{v}_1, \quad \mathbf{x}_2(t) = e^{-2t}\mathbf{v}_2, \quad \mathbf{x}_3(t) = e^{-2t}\left(\mathbf{v}_3 + t(A+2I)\mathbf{v}_3\right).$$

We should be a bit more precise about this last one, it is

$$\mathbf{x}_3(t) = e^{-2t}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}.$$

Problem 38. We are to find a fundamental set of solutions to the equation

$$\mathbf{x}' = \begin{pmatrix} 5 & -1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 1 & 1 & -1 & -3 \\ 0 & -1 & 0 & 7 \end{pmatrix}\mathbf{x}.$$

Call the matrix in question A. We can compute the characteristic polynomial by expanding $A - tI$ along the third column. We get $p_A(t) = (t+1)(t-5)^3$. Thus the eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 5$. It is pretty clear that an eigenvector for A corresponding to $\lambda_1 = -1$ is $\mathbf{v}_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}^T$. We then compute that

$$A - 5I = \begin{pmatrix} 0 & -1 & 0 & 2 \\ 0 & -2 & 0 & 4 \\ 1 & 1 & -6 & -3 \\ 0 & -1 & 0 & 2 \end{pmatrix}$$

has nullspace of dimension two. But

$$(A-5I)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -6 & -6 & -36 & 18 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

has nullspace of dimension three. It is spanned by the three vectors

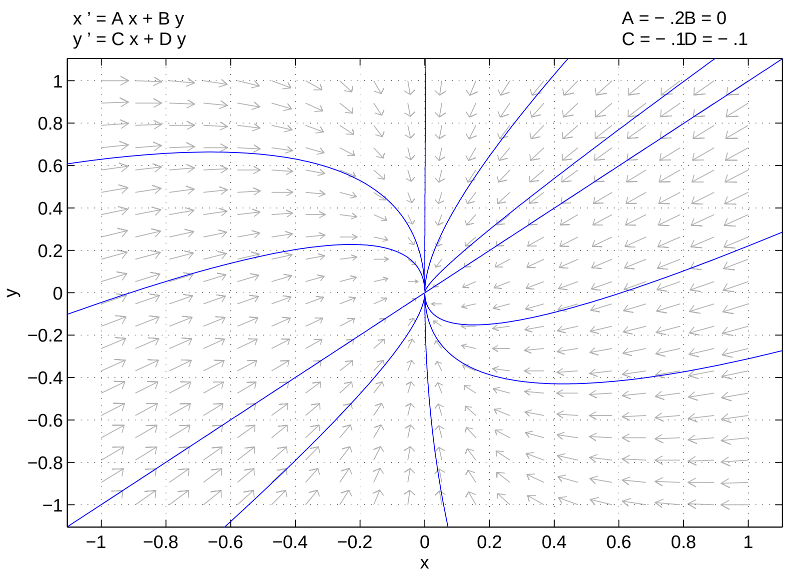
$$\mathbf{v}_2 = \begin{pmatrix} 3 & 0 & 0 & 1 \end{pmatrix}^T, \quad \mathbf{v}_3 = \begin{pmatrix} 1 & -1 & 0 & 0 \end{pmatrix}^T, \quad \mathbf{v}_4 = \begin{pmatrix} 6 & 0 & 1 & 0 \end{pmatrix}^T.$$

We then can compute the other fundamental solutions as

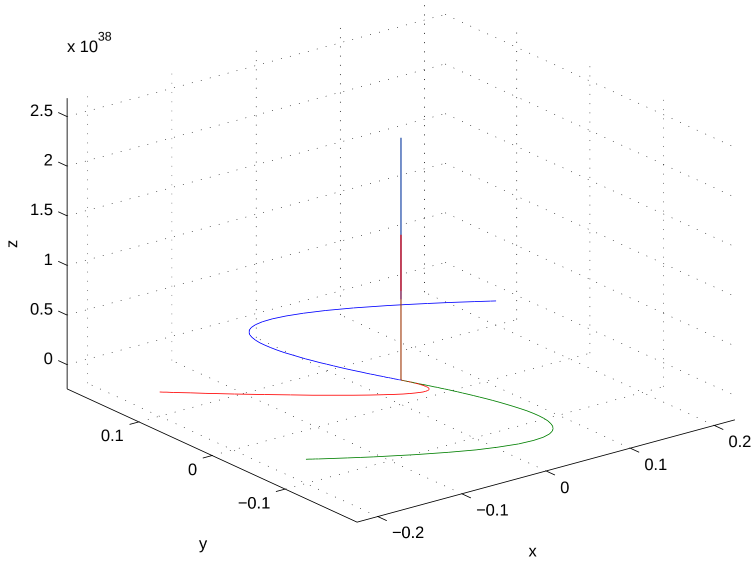
$$\begin{aligned}
\mathbf{x}_2 &= e^{tA}\mathbf{v}_2 = e^{5t}(\mathbf{v}_2 + t(A - 5I)\mathbf{v}_2) \\
&= e^{5t}\begin{pmatrix} 3+2t & 4t & 0 & 1+2t \end{pmatrix}^T \\
\mathbf{x}_3 &= e^{tA}\mathbf{v}_3 = e^{5t}(\mathbf{v}_3 + t(A - 5I)\mathbf{v}_3) \\
&= e^{5t}\begin{pmatrix} 1+t & -1+2t & 0 & t \end{pmatrix}^T \\
\mathbf{x}_4 &= e^{tA}\mathbf{v}_4 = e^{5t}(\mathbf{v}_4 + t(A - 5I)\mathbf{v}_4) \\
&= e^{5t}\begin{pmatrix} 6 & 0 & 1 & 0 \end{pmatrix}^T
\end{aligned}$$

1.2. **Section 9.6.** My solutions are going to get more terse. . .

Problem 6. The matrix has eigenvalues $\lambda_1 = -\frac{1}{10}$ and $\lambda_2 = \frac{-1}{20}$ with eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$ and $\mathbf{v}_2 = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$, respectively. Since both eigenvalues are real and negative, the origin is an asymptotically stable equilibrium point. In fact, it is a stable node. I used `pplane` to draw a phase portrait.

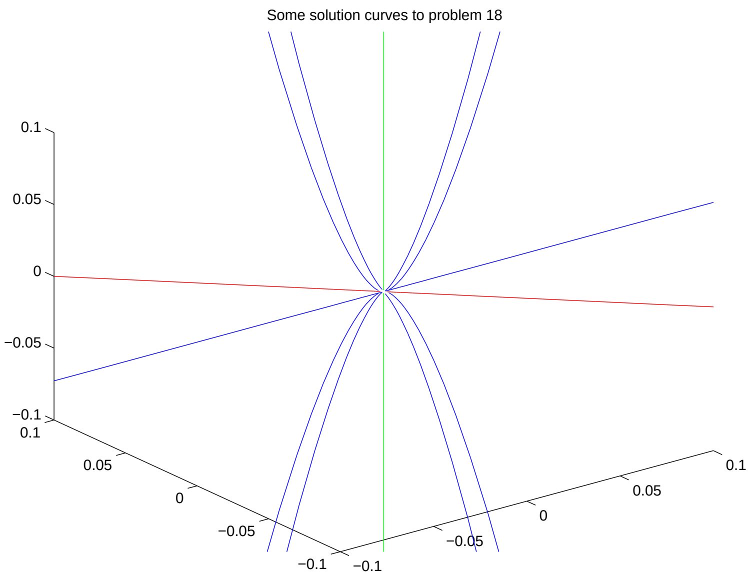


Problem 10. The eigenvalues of this matrix are $3, -1, -2$. Since one of these is positive, the origin is an unstable equilibrium point. I used `odesolve` to plot three "random" initial conditions. (Actually, they are just in "general position". Humans can't do "random" choices.)



You can see that for each of these solution curves the z coordinate explodes.

Problem 18. This matrix has eigenvalues of $1, 2, 3$. The corresponding eigenvectors are $\mathbf{v}_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$, $\mathbf{v}_2 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$, and $\mathbf{v}_3 = \begin{pmatrix} 1 & -1. & 0 \end{pmatrix}^T$, respectively. The picture I got is below. This looks like a three dimensional version of a nodal source. To get a good feel for this, you should rotate it around and zoom in.



1.3. **Section 9.7.**

Problem 9. It is not difficult to check that these are both solutions, and that at $t = 0$ their Wronskian is equal to 1.

Problem 31. This equation has eigenvalues $\pm i$. Both are of algebraic multiplicity two, but geometric multiplicity one. After much work with matrices and (generalized) eigenvectors, we see that the general solution to this equation is

$$x(t) = C_1 \cos(t) + C_2 \sin(t) + C_3 t \cos(t) + C_4 t \sin(t).$$

(Remember, the solution is basically in the first coordinate of the corresponding solution to the vector equation).

Problem 36. This has characteristic equation $t^2 + 4t + 4 = 0$ and thus only one eigenvalue $\lambda = -2$ of algebraic multiplicity two. The geometric multiplicity is one. We find fundamental set of solutions to the corresponding matrix equation to be

$$\mathbf{x}(t) = e^{-2t} \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad \text{and} \quad \mathbf{x}_2(t) = e^{-2t} \begin{pmatrix} 1 + 2t \\ -4t \end{pmatrix}.$$

Thus our general solution is

$$y(t) = C_1 e^{-2t} + C_2 t e^{-2t}.$$

Checking the initial conditions, we see we must take $C_1 = 2$ and $C_2 = 3$, so that $y(t) = 2e^{-2t} + 3te^{-2t}$.

1.4. **Section 9.8.**

Problem 4. We are to solve the inhomogeneous equation $y'(t) = Ay(t) + \mathbf{f}$, where

$$A = \begin{pmatrix} -3 & 10 \\ -3 & 8 \end{pmatrix}, \quad \text{and} \quad \mathbf{f} = \begin{pmatrix} e^{-t} \\ e^{2t} \end{pmatrix}.$$

Following the procedure for such equations, we first solve the associated homogeneous problem. A system of fundamental solutions to $y' = Ay$ is

$$x_1(t) = \begin{pmatrix} 2e^{2t} \\ e^{2t} \end{pmatrix}, \quad \text{and} \quad x_2(t) = \begin{pmatrix} 5e^{3t} \\ 3e^{3t} \end{pmatrix}.$$

Thus we can form a fundamental matrix

$$Y(t) = \begin{pmatrix} 2e^{2t} & 5e^{3t} \\ e^{2t} & 3e^{3t} \end{pmatrix},$$

and thus find a particular solution to our problem.

$$\begin{aligned} x_p(t) &= Y(t) \cdot \int Y(t)^{-1} f(t)\ dt \\ &= Y(t) \cdot \int \begin{pmatrix} 3e^{-2t} & -5e^{-2t} \\ -e^{-3t} & 2e^{-3t} \end{pmatrix} \begin{pmatrix} e^{-t} \\ e^{2t} \end{pmatrix}\ dt \\ &= Y(t) \cdot \int \begin{pmatrix} 3e^{-3t} - 5 \\ -e^{-4t} + 2e^{-t} \end{pmatrix}\ dt \\ &= \begin{pmatrix} 2e^{2t} & 5e^{3t} \\ e^{2t} & 3e^{3t} \end{pmatrix} \begin{pmatrix} -3e^{-3t} - 5t \\ -\frac{1}{4}e^{-4t} + 2e^{-t} \end{pmatrix} \\ &= -\frac{e^{2t}}{4} \begin{pmatrix} 3e^{-3t} + 40t + 40 \\ e^{-3t} + 20t + 24 \end{pmatrix} \end{aligned}$$

The general solution to the inhomogeneous problem is now

$$y(t) = x_p(t) + C_1 x_1(t) + C_2 x_2(t).$$

Problem 7. This is a lot like the last problem. The eigenvalues are $2, 1, 0$. We get corresponding exponential solutions to the associated homogeneous equations as

$$x_1(t) = e^{2t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad x_2(t) = e^{t} \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}, \quad x_3(t) = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}.$$

So, we can find all the necessary parts to make a particular solution to the inhomogeneous equation as follows.

$$Y(t) = [x_1(t), x_2(t), x_3(t)], \quad Y(t)^{-1} f = \begin{pmatrix} 0 \\ 0 \\ \sin(t) \end{pmatrix}, \quad \int Y(t)^{-1} f\ dt = \begin{pmatrix} 0 \\ 0 \\ \cos(t) \end{pmatrix}$$

Hence a particular solution is

$$x_p(t) = Y(t) \int Y(t)^{-1} f(t)\ dt = \begin{pmatrix} \cos(t) \\ -2\cos(t) \\ 0 \end{pmatrix}.$$

The general solution is the sum $x_p + C_1 x_1 + C_2 x_2 + C_3 x_3$.

Problem 17. Given $A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}$ and $y_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, we are to compute e^{tA} and the solution to $y' = Ay, y(0) = y_0$.
The eigenvalues of A are $-1, 2$. From these we build exponential solutions

$$x_1(t) = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix} \quad \text{and} \quad x_2(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}.$$

This allows us to make a fundamental matrix $Y(t)$ and compute that

$$e^{tA} = Y(t) \cdot Y(0)^{-1} = \begin{pmatrix} \dfrac{e^{-t} + 2e^{2t}}{3} & \dfrac{-e^{-t} + e^{2t}}{3} \\ \dfrac{-2e^{-t} + 2e^{2t}}{3} & \dfrac{2e^{-t} + e^{2t}}{3} \end{pmatrix}.$$

Also, we see that the solution to the initial value problem is

$$y(t) = e^{tA} y_0 = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}.$$

Problem 18. This problem is just like the last one. We get that

$$e^{tA} = \begin{pmatrix} -e^{-t} + 2e^{-4t} & -e^{-t} + e^{-4t} \\ 2e^{-t} - 2e^{-4t} & 2e^{-t} - 2e^{-4t} \end{pmatrix},$$

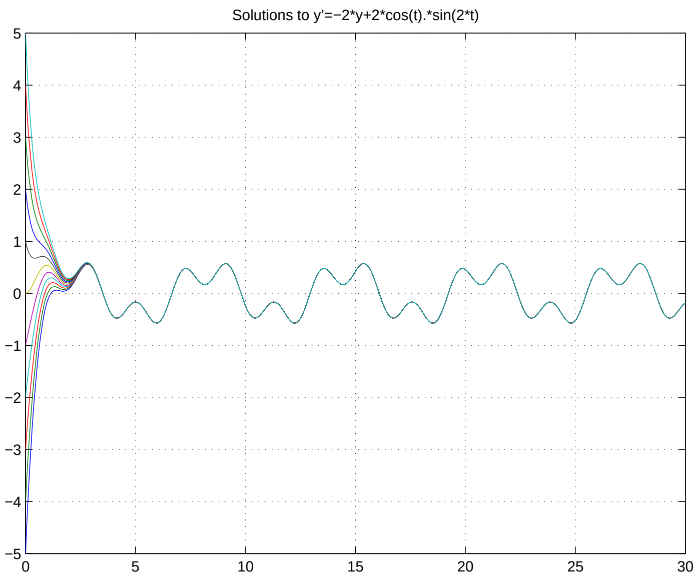
and

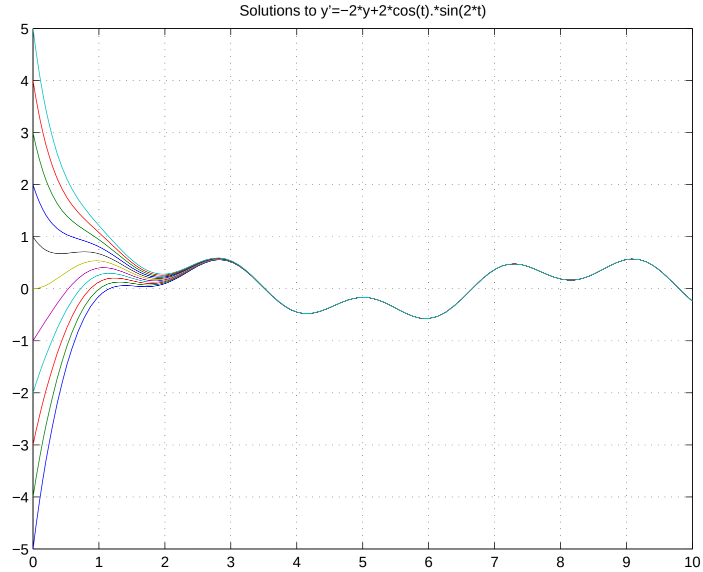
$$y(t) = e^{tA} y_0 = \begin{pmatrix} -e^{-t} + 2e^{-4t} \\ 2e^{-t} - 2e^{-4t} \end{pmatrix}.$$

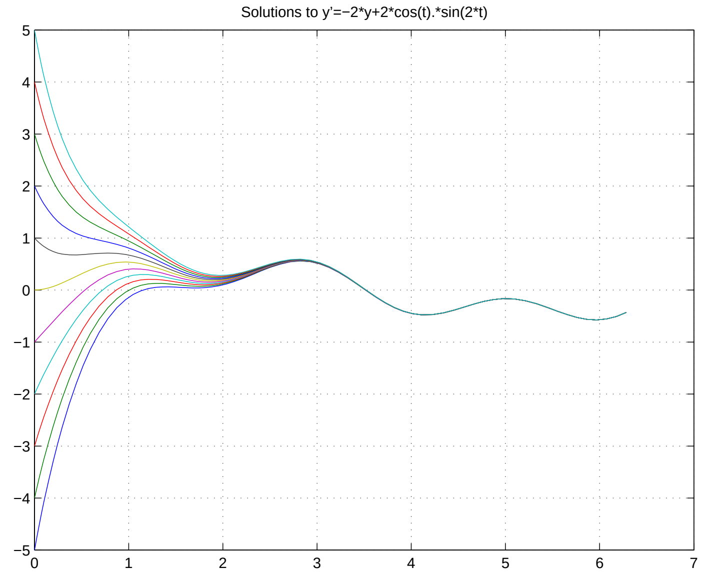
2. Exercises from the Manual

2.1. Chapter 8.

Problem 1. The output of this problem is three pictures, which I'll include below.





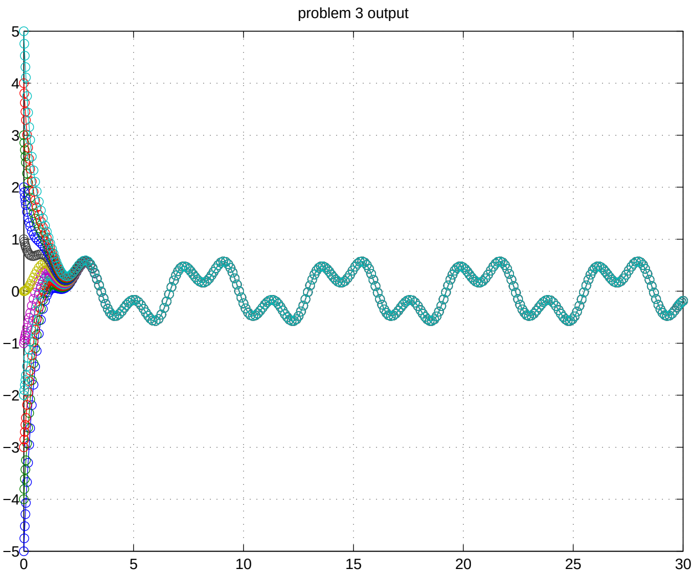


Problem 2. When entering the suggested commands, MATLAB will return the following data.

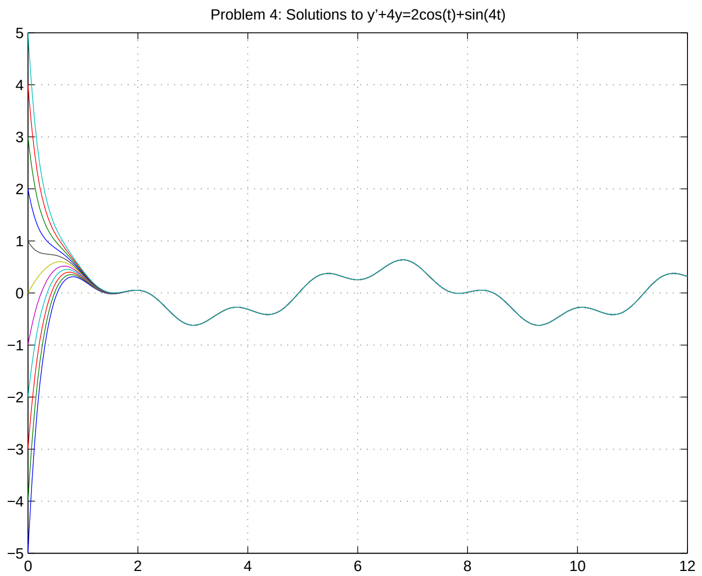
```
>> [t,y]=ode45(@steady,0:.25:3,1);
>> [t,y]
ans =
          0     1.0000
     0.2500     0.7089
     0.5000     0.6797
     0.7500     0.7103
     1.0000     0.6723
```

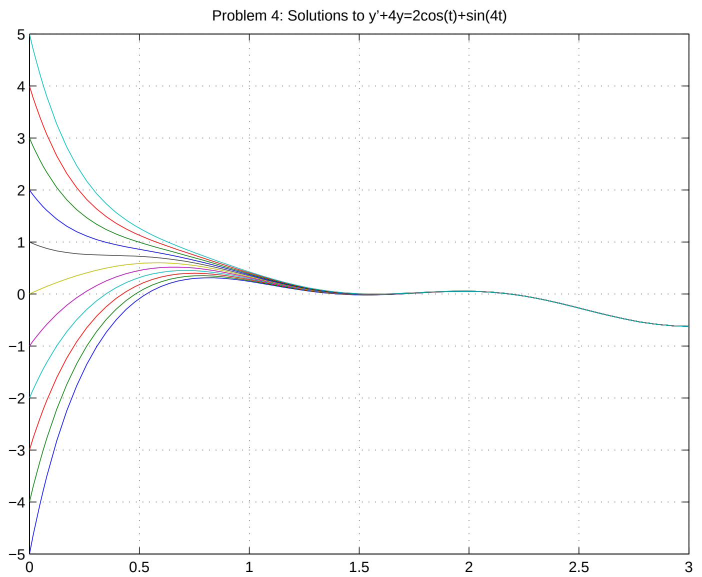
```
1.2500    0.5354
1.5000    0.3544
1.7500    0.2221
2.0000    0.2086
2.2500    0.3159
2.5000    0.4736
2.7500    0.5743
3.0000    0.5317
```

Problem 3. Again, the output is a picture. The fun part is that the pictures are drawn in real time as the computations are done. The picture is below.

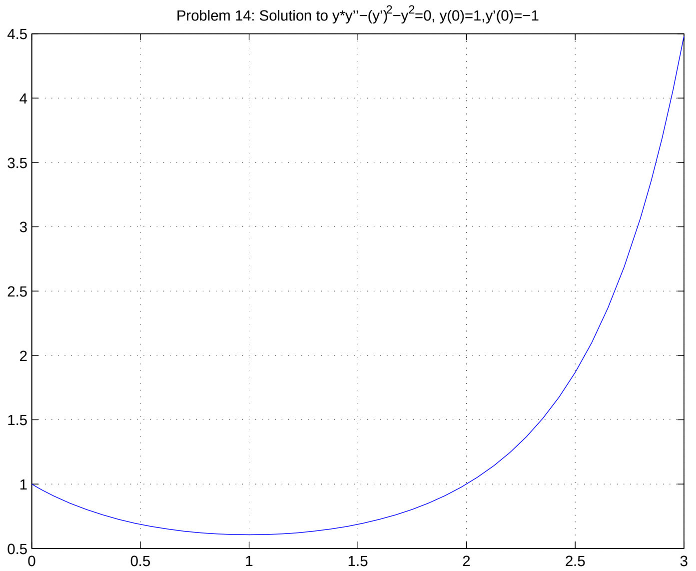


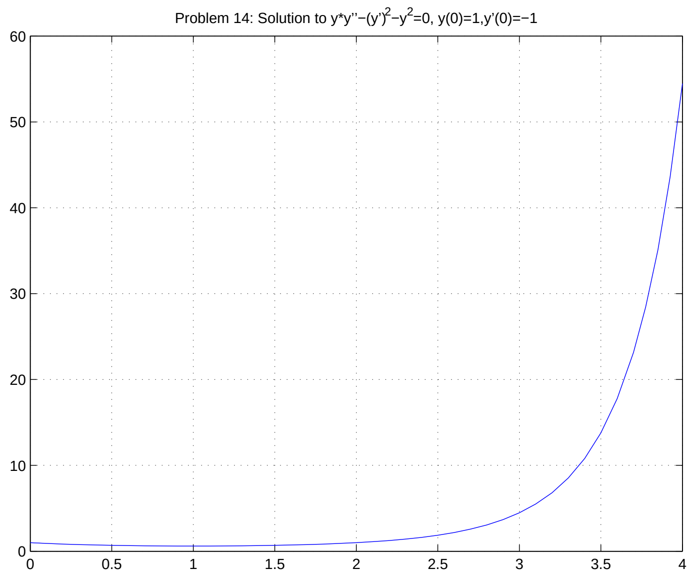
Problem 4. To see the interesting phenomenon, we need an interval for t which is at least as big as $[0, 3]$. To see the steady oscillations, we need as much as $[0, 10]$. The pictures are below.





Problem 14. My output is a pair of graphs. The first is for $0 \leq t \leq 3$. The second is for $0 \leq t \leq 4$ to show the sudden exponential growth.





Problem 18. Again, our output is a lot of pictures, which I include below. I didn't plot the animated ones, because the final versions aren't any different from the 3d plots.