HOMEWORK # 2: MATH 211, SPRING 2005

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Note: This is the last solution set where I will describe the MATLAB *I used to make my pictures.*

1. EXERCISES FROM THE TEXT

1.1. Chapter 2.1. Problem 6. We are to show that $y(t) = 4/(1 + Ce^{-4t})$ is a general solution to the equation $y' = y(4 - y)$, then sketch the particular solutions with $C = 1, 2, 3, 4, 5$ on a computer.

To check that the given expression is a general solution, we compute both sides of the differential equation with it.

$$y'(t) = \frac{-4 \cdot -4Ce^{-4t}}{(1 + Ce^{-4t})^2} = \frac{16Ce^{-4t}}{(1 + Ce^{-4t})^2}$$

and

$$y(t)(4 - y(t)) = \frac{4}{(1 + Ce^{-4t})}\left(4 - \frac{4}{(1 + Ce^{-4t})}\right) = \frac{16Ce^{-4t}}{(1 + Ce^{-4t})^2}.$$

Since these agree, the given expression is a general solution to the given equation. We plot the particular solutions indicated with a MATLAB M-file.

`textch2_1prob6.m`

```
t=linspace(-2.5,2.5,1000);
Y=[];
for C=1:5
    Y=[Y; 4./(1+C*exp(-4*t))];
end
plot(t,Y)
axis([-2.5,2.5,-0.5,4.5])
grid on
legend('C=1','C=2','C=3','C=4','C=5',2)
xlabel('t')
ylabel('y')
title('Solutions to y''=y(4-y)')
shg
```

The output graph is the following.

Solutions to y'=y(4−y)
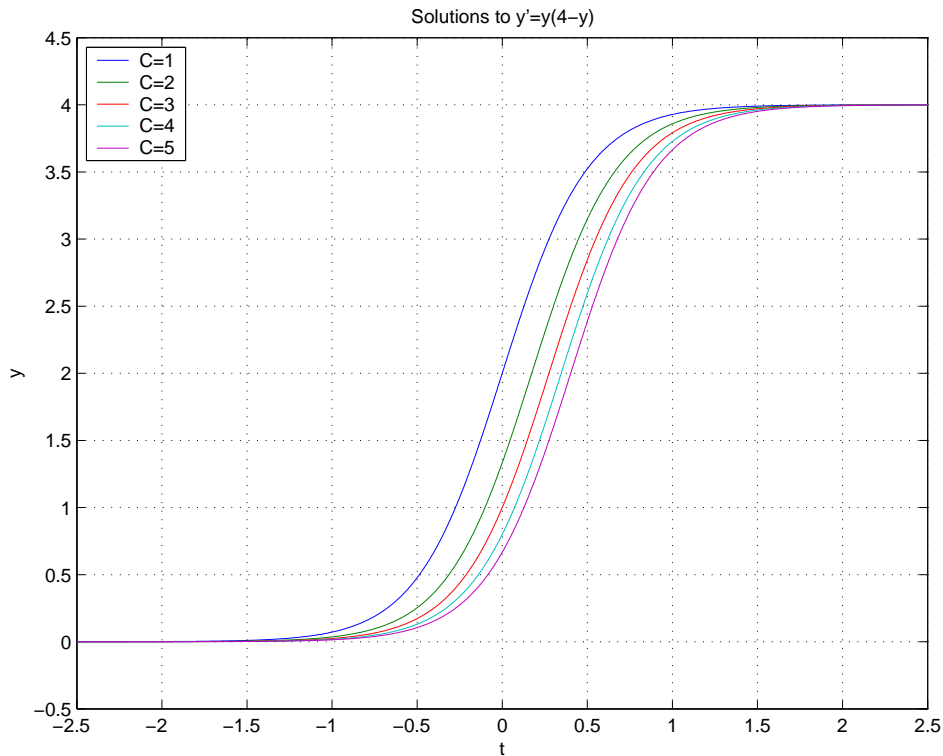
Problem 7. Note that for exercise 6, $y = 0$ is a solution because then $y'(t) = 0$ and $y(t)(4 - y(t)) = 0$. However, no value of C in the general solution can produce this particular solution because the fraction in question can never be zero–its numerator has no roots.

Problem 8. This problem has three parts.

    a: Use implicit differentiation to show $t^2 + y^2 = C^2$ defines solutions to the differential equation $t + yy' = 0$.

        If we differentiate the given expression implicity, we obtain $2t + 2y \cdot y' = 0$. Upon dividing by 2, we get the result.

    b: Solve $t^2 + y^2 = C^2$ for y to provide explicit solutions. Show that these functions are also solutions to the differential equation in (a).

        The algebraic manipulation yeilds $y(t) = \pm\sqrt{C^2 - t^2}$. Note that this is a pair of solutions for each value of C. We differentiate these to find $y'(t) = \mp\dfrac{t}{\sqrt{C^2 - t^2}}$. This allows us to check the differential equation as follows:

$$t + y(t)y'(t) = t + \left(\pm\sqrt{C^2 - t^2}\right) \cdot \left(\mp\frac{t}{\sqrt{C^2 - t^2}}\right) = t - t = 0.$$

    c: Discuss the interval of existence for each of the solutions in part (b).

Fix a value of C, and consider the solutions $y(t) = \pm\sqrt{C^2 - t^2}$. This function is only well-defined for those t which satisfy $C^2 - t^2 \geq 0$. This means that we must restrict at least to the interval $-C \leq t \leq C$. However, we also require that our solution is continuously differentiable. So we exclude the endpoints, too. Thus, the interval of existence for these two solutions is $(-C, C)$.

1.2. **Chapter 2.2. Problem 10.** We are to find a general solution to the equation $xy' - y = 2x^2y$.

To do so by separation of variables, we must first rearrange the equation to read $\dfrac{y'}{y} = \dfrac{2x^2 + 1}{x}$. Then we may integrate with respect to x to find

$$\ln|y(x)| = \int \frac{y'(x)}{y(x)} \, dx = \int \frac{2x^2 + 1}{x} \, dx$$
$$= \int \left(2x + \frac{1}{x}\right) \, dx = x^2 + \ln|x| + C.$$

So $\ln|y(x)| = x^2 + \ln|x| + C$ is our implicit solution. We exponentiate both sides to find

$$|y(x)| = C|x|e^{x^2}.$$

Whatever sign ambiguities come from having the absolute value bars can be absorbed into the constant in this case, so we obtain the following explicit general solution: $y(x) = Cxe^{x^2}$.

Problem 16. We are to find the exact solution to the initial value problem $y' = e^{x+y}$, $\quad y(0) = 0$ and give the interval of existence.

We separate the variables as follows: $e^{-y}y' = e^x$, then integrate both sides to get

$$-e^{-y} - (-1) = \int_0^y e^{-u} \, du = \int_0^x e^w \, dw = e^x - 1.$$

We rearrange to obtain the explicit solution $y(x) = -\ln(2 - e^x)$. (This one I had to double-check to believe. It just felt funny.) In order for the equation to make sense, we need to restrict to those x for which $2 - e^x \geq 0$. That is, we must restrict to $(-\infty, \ln 2]$. Just as before, we need to also assure that our function is continuously differentiable, so we remove the endpoint. The interval of existence is $(-\infty, \ln 2)$.

1.3. **Chapter 2.3. Problem 12.** We are given that a 0.2 kg mass is released from rest at 50 m, and that it falls against an air resistance of $R(v) = -0.1v$, where v is the velocity. What is the velocity upon impact with the ground?

There are two ways to go about this problem. The straightforward way is to write an initial value problem for the velocity and solve it. Then, integrate with respect to time, t, to find the position. We then solve for the time it takes the mass to hit the

ground (numerically, the equation is nasty), and then evaluate the velocity at this time to get an answer.

The clever way is as follows. The velocity is $v = dx/dt$. By the chain rule, the acceleration is

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx} \cdot v.$$

We reinterpret this as saying that *as a function of x*, we can write

$$dx = \frac{v\, dv}{dv/dt}.$$

What we shall do is integrate both sides. The left hand side will give us the total change in position, -50 meters, and the right hand side we need to figure out.

This feels a bit slippery at first, but it is ok. It really depends upon being able to write the quantity dv/dt as a function of v. But this is exactly what our equation of motion will tell us.

Using Newton's laws, we see that the motion of the mass is described by the initial value problem

$$\frac{2}{10}\frac{dv}{dt} = -\frac{2}{10}g - \frac{1}{10}v, \quad v(0) = 0.$$

where $g = 9.8 m/s^2$ is the gravitational acceleration constant near the earth. Working with the manipulation above, we find that

$$-50 = \int_{50}^{0} dx = \int_{v=0}^{v_f} \frac{v\, dv}{dv/dt} = \int_{v=0}^{v_f} \frac{v\, dv}{-g - v/2},$$

where v_f is the final (impact) velocity that we seek. I'll clean up the negative signs and do some algebra

$$50 = \int_{0}^{v_f} \frac{2v}{2g + v}\, dv = 2\int_{0}^{v_f} \frac{2g + v - 2g}{2g + v}\, dv,$$

equivalently,

$$25 = \int_{0}^{v_f} \left(1 - \frac{2g}{2g + v}\right)\, dv = v_f - 2g\ln|2g + v_f| - (0 - 2g\ln|2g + 0|).$$

Cleaning up a bit, we get the following implicit equation for the impact velocity

$$25 = v_f - 2g\ln|2g + v_f| + 2g\ln|2g|.$$

This is the kind of equation that needs to be solved numerically. We do so with the MATLAB command

```
v=solve('25=v - 2*9.8*log(abs(2*9.8+v)) +2*9.8*log(abs(2*9.8))')
```

to find $v_f = -17.3401\ m/s^2$. Note that MATLAB returns three answers and we need to pick the correct one.

1.4. **Chapter 2.4. Problem 4.** We are asked to find the general solution to the equation $y' + 2ty = 5t$.

We proceed by finding an integrating factor. We are looking for a function $u(x)$ which satisfies $u'/u = 2t$. Thus $u = e^{t^2}$ will do. Thus, we find that

$$e^{t^2} y(t) = \int (e^{t^2} y)' \ dt = \int e^{t^2} (y' + 2ty) \ dt = \int 5t^2 e^{t^2} dt = \frac{5}{2} e^{t^2} + C.$$

This leads to the general solution of $y(t) = \frac{5}{2} + Ce^{-t^2}$.

Problem 13. This problem has two parts.

a: Use the technique of integrating factors to find the general solution to the equation $y' + y\cos(x) = \cos(x)$.

We must find a function $u(x)$ such that $u'/u = \cos(x)$. Integrating, $u(x) = e^{\sin(x)}$ will do. Then we find that

$$e^{\sin(x)} y = \int (e^{\sin(x)} y)' \ dx = \int e^{\sin(x)} (y' + cos(x)y) \ dx$$
$$= \int e^{\sin(x)} \cos(x) \ dx = e^{\sin(x)} + C.$$

Thus our general solution is $y(x) = 1 + Ce^{-\sin(x)}$.

b: Use separation of variables to solve the same equation. Discuss any discrepancies between the two results.

We separate variables to find $y'/(1 - y) = \cos(x)$. Integrating this, we obtain $-\ln|1 - y| = \sin(x) + C$. This is an implicit equation for the solution. At first blush, this seems different, but upon rearranging things, (and folding any ambiguity of sign into the arbitrary constant) we get the same result as in (a),

$$y(x) = 1 + Ce^{-\sin(x)}.$$

2. Exercises from the Manual

2.1. **Chapter 2. Problem 2.** We are to solve $y' = t(y + 1)$ with $y(0) = 1$ over the interval $[0, 2]$, and use `ezplot` to make a plot of the solution.

First, we solve the equation using separation of variables.

$$\frac{dy}{dt} = t(y + 1)$$

leads to

$$\int \frac{dy}{y + 1} = \int t \ dt.$$

Actually, to stick with the initial conditions, we use this version

$$\int_1^y \frac{du}{u + 1} = \int_0^t \ dw.$$

Performing the integration, we get

$$\ln|y+1| - \ln|2| = t - 0 = t.$$

This is an implicit equation for our solution. Rearranging things, we find $y = 2e^t - 1$ (we choose the positive possibility to match our initial condition). Then we graph this with the `ezplot` command. My MATLAB command line was:

```
ezplot('2*exp(x)-1',[0,2,0.5,14.5]); xlabel('t'); title('Solution
to y''=t(y+1), y(0)=1')
```

(You need to play a bit to find the right interval for y.) This produced the following picture.
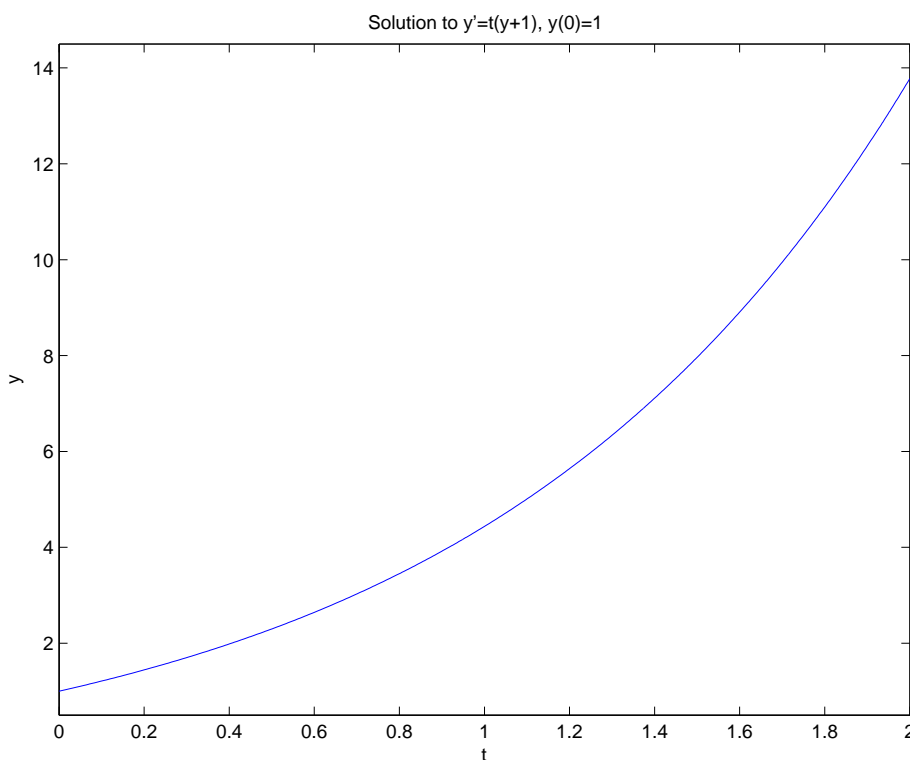
Problem 8. We are to solve $(1 + 3y^2)y' = \cos(t), \quad y(0) = 1$ and use the implicit version of `ezplot` to graph the solution. This equation already has its variables separated, so we integrate

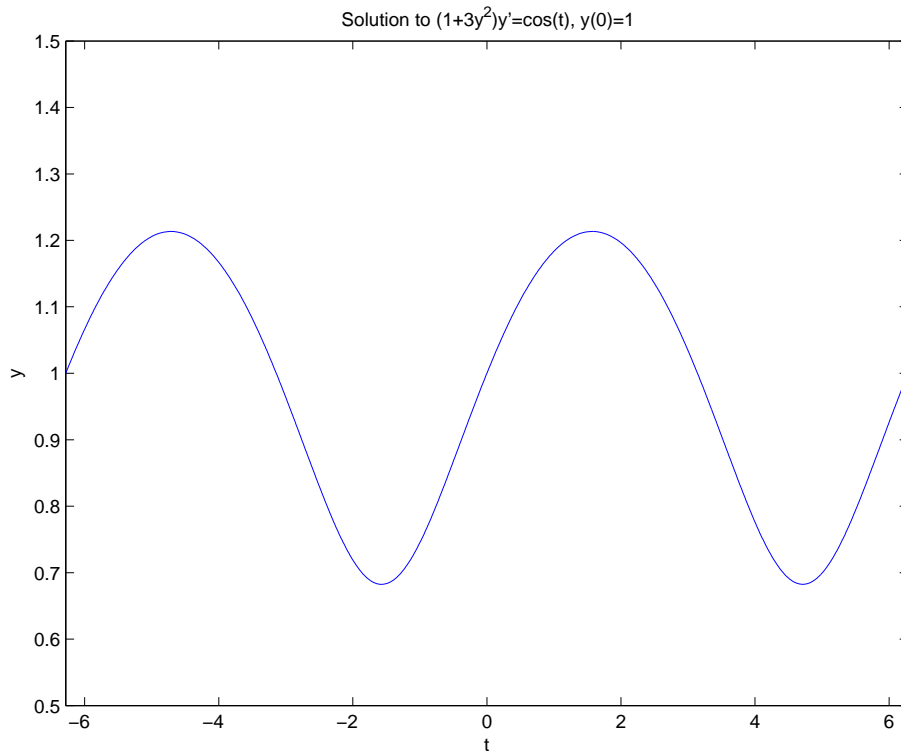$$\int_1^y (1 + 3u^2) \, du = \int_0^t \cos(w) \, dw.$$

This yields

$$y + y^3 - 2 = \sin(t).$$

As we are stuck with this implicit equation, we plot it as follows in MATLAB:

```
ezplot('y+y^3-sin(x)-2',[-2*pi,2*pi,0.5,1.5]);xlabel('t');
title('Solution to (1+3y^2)y''=cos(t), y(0)=1')
```

We get the following.

Problem 21. We are to solve the initial value problem $y' = y\sin(t), \quad y(0) = 1$ and write a script M-file to plot the solution over the interval $[-2\pi, 2\pi]$. For the solution, we separate variables and integrate:

$$\int_1^y \frac{du}{u} = \int_0^t \sin(w) \ dw$$

to get

$$\ln|y| = -\cos(t) + 1.$$

Note that we are interested in a problem which has positive values of y at the initial condition, so we can solve for the explicit expression

$$y = e^{1-\cos(t)}.$$

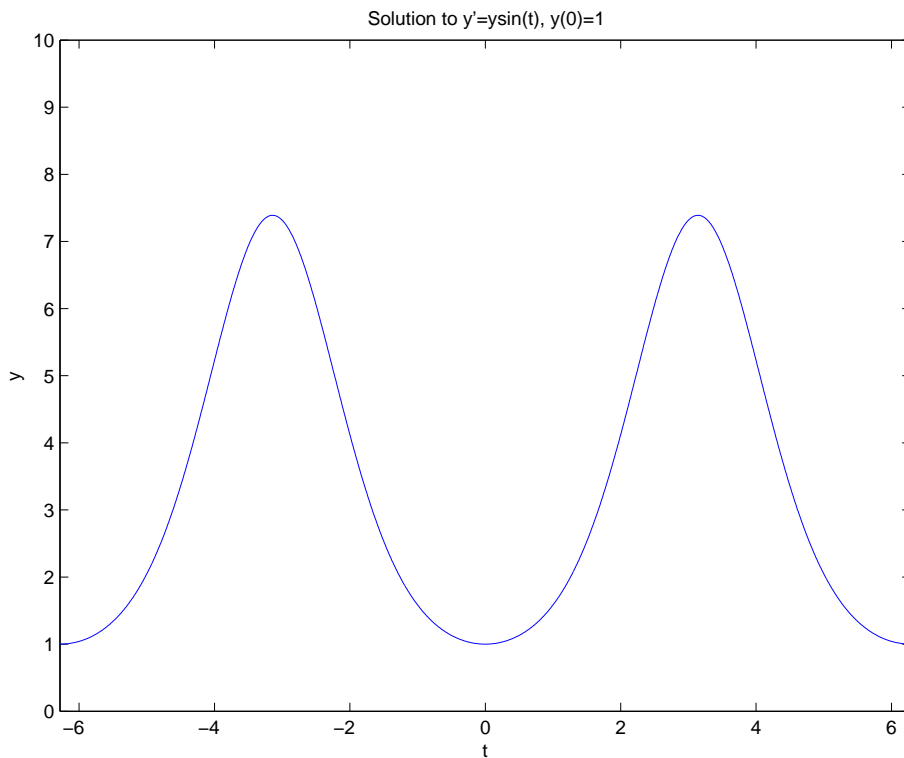I wrote an M-file titled `manualch2prob21.m` with the following contents.

```
t=linspace(-2*pi,2*pi,1000);
y=exp(1-cos(t));
plot(t,y)
axis([-2*pi,2*pi,0,10])
```

```
xlabel('t')
ylabel('y')
title('Solution to y''=ysin(t), y(0)=1')
shg
```

Running it produced the following picture,

2.2. **Chapter 3. Problem 6.** We are to plot solution curves to the equation $x' = 1 - t^2 + \sin(tx)$ with initial values $x = -3, -2, -1, 0, 1, 2, 3$ at $t = 0$. This equation is not very easy to solve, so we just use `dfield` to plot these. A reasonable window to show the "important features" of the graphs is $\leq t \leq, \leq x \leq$. My picture is below. (Soon, I'll figure out how to just print the picture, and not all the extraneous stuff.)

$x' = 1 - t^2 + \sin(t\,x)$