HOMEWORK # 3 SOLUTIONS

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1. Exercises from the text

1.1. **Chapter 2.4. Problem 32** We are to use variation of parameters to find the general solution to $y' + \frac{2}{x}y = 8x$. The associated homogeneous equation is $y' + \frac{2}{x}y = 0$. This is separable, and can be rewritten as $y'/y = -2/x$. We integrate this to find $y = x^{-2}$. So, by the method of variation of parameters, we posit a solution to the original equation of the form $y = v(x) \cdot x^{-2}$, and look for $v(x)$.

Substituting this expression into the original equation we find

$$8x = \frac{v'}{x^2} - \frac{2v}{x^3} + \frac{2}{x}\frac{v}{x^2} = \frac{v'}{x^2}.$$

This is a separable equation which can be rewritten as $v' = 8x^3$. Integrating, we find $v = 2x^4 + C$, and hence the general solution to the original equation is given by

$$y = v(x)x^{-2} = 2x^4 + \frac{C}{x^2}.$$

Problem 38 We are to find the general solution to $y' + y = e^t$ and then specialize to find the specific solution which satisfies the initial condition $y(0) = 1$.

The associated homogeneous equation is $y' + y = 0$, which has solution $y = e^{-t}$. So we search for a solution of the form $y = v(t)e^{-t}$. Substituting into the original equation, we get

$$e^t = y' + y = v'e^{-t} - e^{-t}v + e^{-t}v = v'e^{-t}.$$

Again, this yields a separable equation, and we separate the variables to get $v' = e^{2t}$. This means that $v(t) = e^{2t}/2 + C$, and hence our general solution is

$$y(t) = \frac{e^t}{2} + Ce^{-t}.$$

In order to satisfy the initial condition, we solve for C in $1 = y(0) = 1/2 + C$. Hence the particular solution asked for is $y(t) = \frac{e^t + e^{-t}}{2}$.

Problem 41 Again, we are to find the general solution to $(t^2+1)x' + 4tx = t$ by variation of parameters and then pick out the particular solution which satisfies the initial condition $x(0) = 1$.

The associated homogeneous equation is $(t^2+1)x' + 4tx = 0$. We separate the variables to obtain the equation

$$\frac{x'}{x} = \frac{-4t}{t^2+1}.$$

This takes some work to integrate the left hand side. I used the trig substitution $t = \tan\theta$ and then the substitution $u = \cos\theta$. This is not the most efficient way, but it works. Anyway, the solution is

$$x(t) = \frac{1}{(t^2+1)^2}.$$

Hence, we look for solutions to the original equation of the form $x(t) = v(t)\cdot(t^2+1)^{-2}$. Substituting this into the original equation we see that

$$\begin{aligned} t &= (t^2+1)\left((t^2+1)^{-2}v'(t) - 4tv(t)(t^2+1)^{-3}\right) + 4tv(t)(t^2+1)^{-2} \\ &= v'(t)(t^2+1)^{-1}. \end{aligned}$$

This equation is separable and can be rearranged to read

$$v'(t) = t(t^2+1).$$

This has general solution $v(t) = \frac{(t^2+1)^2}{4} + C$. So we see that the general solution to our original equation is

$$y(t) = \frac{1}{4} + \frac{C}{(t^2+1)^2}.$$

Finally, to satisfy the condition $x(0) = 1$, we must have $C = 3/4$. Thus the particular solution asked for is

$$y(t) = \frac{1}{4} + \frac{3}{4(t^2+1)^2}.$$

1.2. **Chapter 2.4. Problem 7** Let $V(t)$ be the volume (in gallons) of the lake t minutes after the process starts, and let $x(t)$ be the amount (in pounds) of pollutant in the lake at time t.

a: We note that for this part, the equation for volume is $V(t) = 100 + 6t + 4t - 8t = 100 + 2t$. We also write the differential equation for the pollutant

$$\begin{aligned} x'(t) &= (6\ gal/min)(0.5\ lb/gal) - (8\ gal/min)(\frac{x(t)\ lb}{V(t)\ gal}) \\ &= 3 - \frac{8}{100+2t}x. \end{aligned}$$

This equation is linear. We rearrange it to read $x' + \frac{8}{100+2t}x = 3$, and solve it using an integrating factor. The integrating factor must satisfy

$$\frac{u'}{u} = \frac{8}{100+2t}.$$

So we solve, obtaining $u(t) = (100+2t)^4$. Now, we apply the rest of the method to find

$$\begin{aligned} ux &= \int (ux)'\ dt = \int (ux' + u'x)\ dt = \int \left((100+2t)^4 x' + 8(100+2t)^3 x\right)\ dt \\ &= \int (100+2t)^4 \left(x' + \frac{8}{100+2t}x\right)\ dt = \int (100+2t)^4 \cdot 3\ dt \\ &= \frac{3}{10}(100+2t)^5 + C. \end{aligned}$$

We then solve for x,

$$x(t) = \frac{3}{10}(100+2t) + \frac{C}{(100+2t)^4}.$$

To satisfy the initial condition, we solve for C.

$$0 = x(0) = \frac{3}{10}(100) + \frac{C}{(100)^4}.$$

This means $C = -3 \cdot 10^9$. (That's right, negative three billion.) So, we can write our solution out as

$$x(t) = \frac{3}{10}(100+2t) - \frac{3 \cdot 10^9}{(100+2t)^4},$$

and evaluate at $t = 10$ to get $x(10) = 36 - 30 \cdot (5/6)^4 \approx 21.5324\ lbs$.

b: If we restart and turn off pipe A, how long will it take to cut the amount of pollutant in half? Let A be the amount of pollutant at the (new) starting time. We also have $V(0) = 120$, with $V(t) = 120 - 4t$ as the new volume equation. Our new differential equation for the amount of pollutant is

$$x'(t) = -\frac{8x(t)}{V(t)} = \frac{-8x}{120 - 4t}.$$

This equation is separable, and we rearrange it to read

$$\frac{x'}{x} = \frac{-8}{100 - 4t}.$$

This can be integrated to $\ln|x| = 2\ln|120 - 4t| + C$. That is, $x(t) = C(100 - 4t)^2$. Clearly, to meet the initial condition, we must have $C = A/120^2$. So,

$$x(t) = \frac{A}{120^2}(120 - 4t)^2.$$

We are looking for the time so that $x(t) = A/2$. So we are to solve

$$\frac{A}{2} = \frac{A}{120^2}(120 - 4t)^2.$$

Note that the actual value of A is unimportant here, so we can avoid roundoff errors that would occur by cancelling the factor of A now. Anyway, the solutions are $t = 30 \pm 15\sqrt{2}$. The best one for us is the first one, which is about 8.7868 *min*.

1.3. **Chapter 2.4. Problem 5** Compute the total differential of $F(x, y) = xy + \arctan(y/x)$.

We see that

$$P = \partial F/\partial x = y + \frac{1}{1 + (y/x)^2} \cdot \frac{-y}{x^2} = y - \frac{y}{y^2 + x^2}$$

and that

$$Q = \partial F/\partial y = x + \frac{1}{1 + (y/x)^2} \cdot \frac{1}{x} = x + \frac{1}{x + y^2/x}.$$

Thus the total differential of F is

$$dF = P\ dx + Q\ dy = \left(y - \frac{y}{y^2 + x^2}\right)\ dx + \left(x + \frac{1}{x + y^2/x}\right)\ dy.$$

For the next three exercises, say whether or not the equation is exact. If it is, solve it.

Problem 10 $(1 - y \sin x)\ dx + (\cos x)\ dy = 0$

We check that $\partial P/\partial y = -\sin x$, and $\partial Q/\partial x = -\sin x$, so this equation is exact.

To solve, we proceed as follows.

$$F(x, y) = \int P\ dx = \int (1 - y \sin x)\ dx = x + y \cos x + C(y).$$

To compute $C(y)$, we differential with respect to y:

$$\partial F/\partial y = 0 + \cos x + C'(y)$$

and compare to $Q = \cos x$. We learn that $C'(y) = 0$. Thus our solution is $F(x, y) = x + y \cos x = C$, where C is an arbitrary constant.

Problem 14 $\frac{dy}{dx} = \frac{x}{x-y}$

This should be rearranged to $x\ dx - (x - y)\ dy = 0$. We compute that $\partial P/\partial y = 0$ and $\partial Q/\partial x = -1$. So this equation is not exact.

Problem 16 $\frac{2u}{u^2+v^2}\ du + \frac{2v}{u^2+v^2}\ dv = 0$

We check that $\partial P/\partial v = -\frac{-4uv}{(u^2+v^2)^2}$, and $\partial Q/\partial u = -\frac{-4uv}{(u^2+v^2)^2}$. Hence, this equation is exact. So we solve by the standard procedure.

$$F(x, y) = \int P\ du = \int \frac{2u}{(u^2 + v^2)^2}\ du = \ln(u^2 + v^2) + C(v).$$

To find $C(v)$, we take the derivative with respect to v.

$$Q = \frac{\partial F}{\partial v}\left[\ln(u^2 + v^2) + C(v)\right] = \frac{2v}{u^2 + v^2} + C'(v).$$

So it suffices to take $C(v) = 0$. Therefore, our solution is

$$F(u, v) = \ln(u^2 + v^2) = C,$$

where C is an arbitrary constant.

2.1. **Chapter 3.** In this group of exercises, we consider a lake with volume $V = 100\ km^3$ which is fed by a river at the rate of $r_i\ km^3/year$ and drained by another river in a way to keep the total volume of water in the lake constant. There is a factory which pollutes the lake at a rate of $p\ km^3/year$, too. Thus the river out flows at a rate of $p + r_i\ km^3/year$. Let $x(t)$ be the volume of pollutant in the lake at time t years and let $c(t) = x(t)/V$ be the concentration of pollutant in the lake at time t years. Note that this concentration is unitless.

Problem 11 Show that with perfect and immediate mixing of pollutant into the lake, the concentration c satisfies the following differential equation $c' + ((p+r_i)/V)c = p/V$.

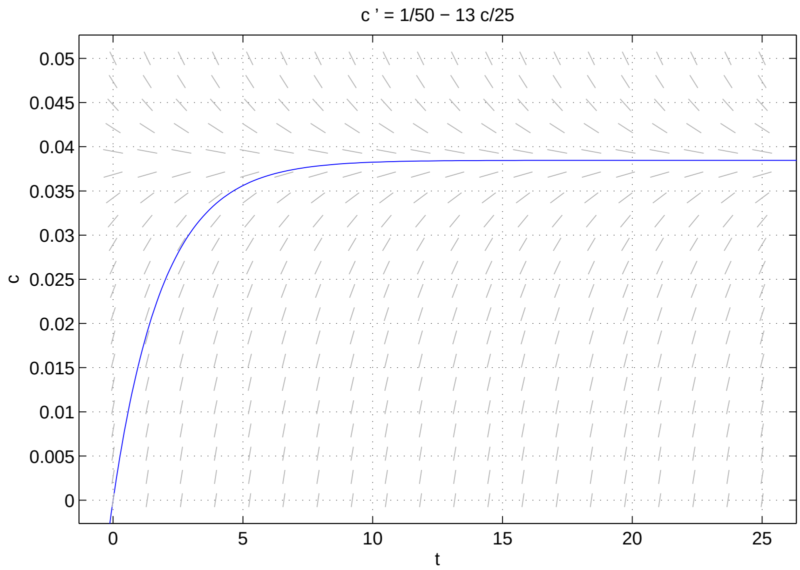
First, we work out the equation describing the amount of pollutant in the lake. The only way in is from the factory and only way out is through the second river, so
RATE IN $= p\ km^3/year$.
RATE OUT = (concentration)(flow rate) $= c(t)(p + r_i\ km^3/year)$.

This means that $x'(t) = p - c(t)(p + r_i)\ km^3/year$. To achieve our goal, we need only divide by the total volume, since $c(t) = x(t)/V$. This gives the desired result after a small amount of rearranging. Note that $c'(t)$ has units of $(years)^{-1}$.

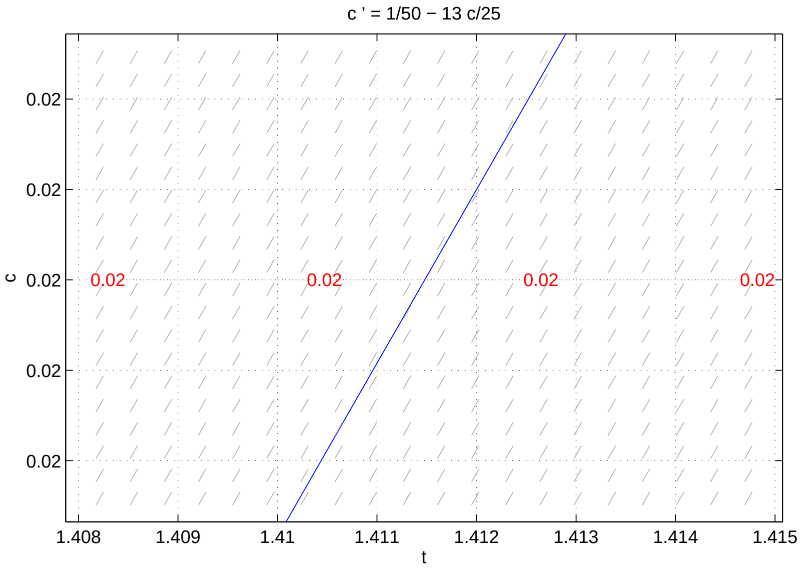
Problem 12 Suppose that $r_i = 50$ and $p = 2$. Thus our equation is

$$c'(t) + \frac{13}{25}c(t) = \frac{1}{50}.$$

a: Assume that $c(0) = 0$. Use `dfield` to plot the solution.



b: Approximately how long until the concentration of pollutant is 2%? (Recall that percent is exactly the way to talk about a unitless thing like concentration when you divide it by 100.) We zoom in with `dfield`. Since `dfield` has trouble keeping track of decimal digits on the vertical axis, we also plot the level curve $c = .02$. From this we approximate that it will take about $t \approx 1.4115$ *years*.



c: What is the limiting concentration? And how long does it take to reach a concentration of 3.5%? I'll drop the graphs for this, but exactly the process as in the last part helps us to find that the limiting concentration is approximately 3.8642% and it takes approximately 4.6307 *years* to reach a concentration of 3.5%. It is helpful to use the level curves again.

Problem 13 So now we start over, where the factory stops operation at $t = 0$, but has left us with an initial concentration of pollutant of 3.5%. That is $c(0)/100 = .035$. How long does it take to get back down to the safe level of 2%?

This new problem has a slightly different equation. It is

$$c'(t) = -\frac{r_i}{V} c(t).$$

So now we have an initial value problem of

$$c'(t) = -\frac{1}{2} c(t), c(0) = .035,$$

and we are asked to find how long it takes for the solution to reach $c(t) = .02$. Fortunately, this equation is separable, and we can proceed directly. The solution curve is $c(t) = .035e^{-t/2}$. Thus it is not difficult to compute that it takes $t = -2\ln(20/35) \approx 1.1192\ years$.