

HOMEWORK # 4 SOLUTIONS

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1. EXERCISES FROM THE TEXT

1.1. Chapter 2.7.

Problem 2 This equation is in normal form and the right hand side has a continuous partial derivative with respect to y as long as we avoid points where $y \leq 0$. Since our initial condition $y(4) = 0$ is at the boundary of the good and bad regions, the theorem does not apply.

Problem 4 This equation is also in normal form. The right hand side has continuous partial derivative with respect to ω at all points (s, ω) . Thus, we have a unique solution to any initial value problem here, including the given one.

Problem 7 This problem concerns the differential equation $ty' - y = t^2 \cos(t)$, and the initial value problem obtained by also requiring $y(0) = -3$.

- First we find the general solution and graph several members of the family on a direction field. The equation is linear, so we may use either of the methods we learned earlier this term. I'll use variation of parameters (just to keep sharp). The associated homogeneous equation is $ty' - y = 0$. As expected, this is separable and we rearrange it to read $y'/y = 1/t$. We integrate to find $\ln |y(t)| = \ln |t| + C$ and then exponentiate to get $y(t) = Ct$. Notice that any potential ambiguities of sign have disappeared into the constant.

Following the variation of parameters scheme, we look for a solution of the type $y(t) = v(t) \cdot t$. If such a function satisfies the differential equation, we deduce

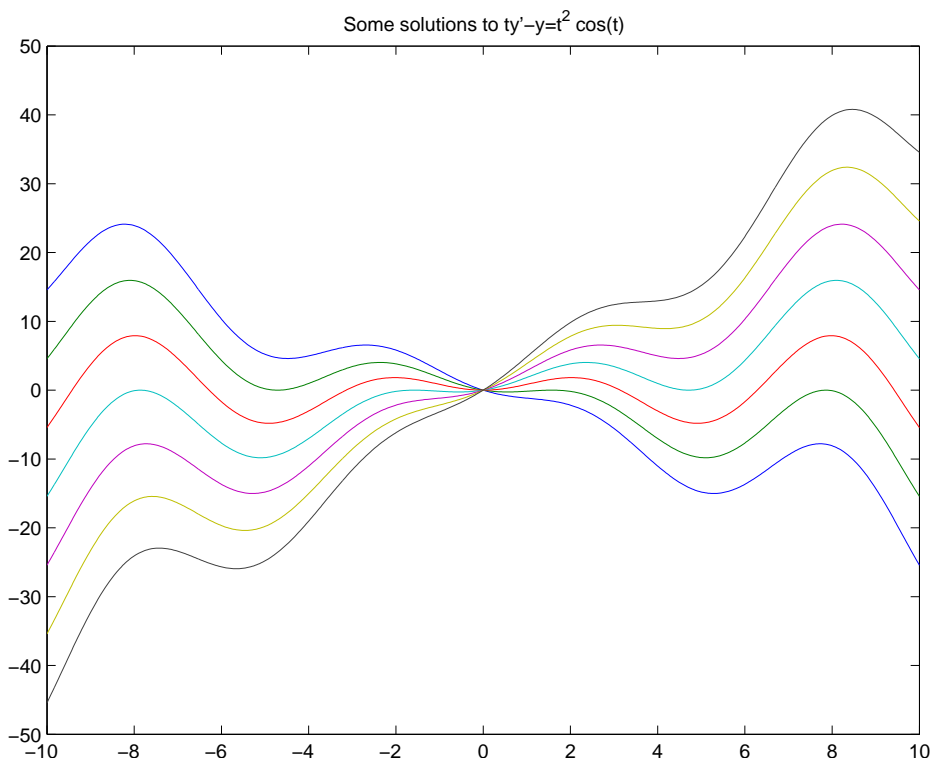
$$t(v(t) + v'(t) \cdot t) - v(t) \cdot t = t^2 \cos(t).$$

We clean this up a bit to read $v'(t)t^2 = t^2 \cos(t)$. If this is going to happen, we must have $v'(t) = \cos(t)$ and hence $v(t) = \sin(t) + C$.

Putting this all together, we get a general solution of

$$y(t) = t \sin(t) + Ct.$$

I have graphed several of these below.

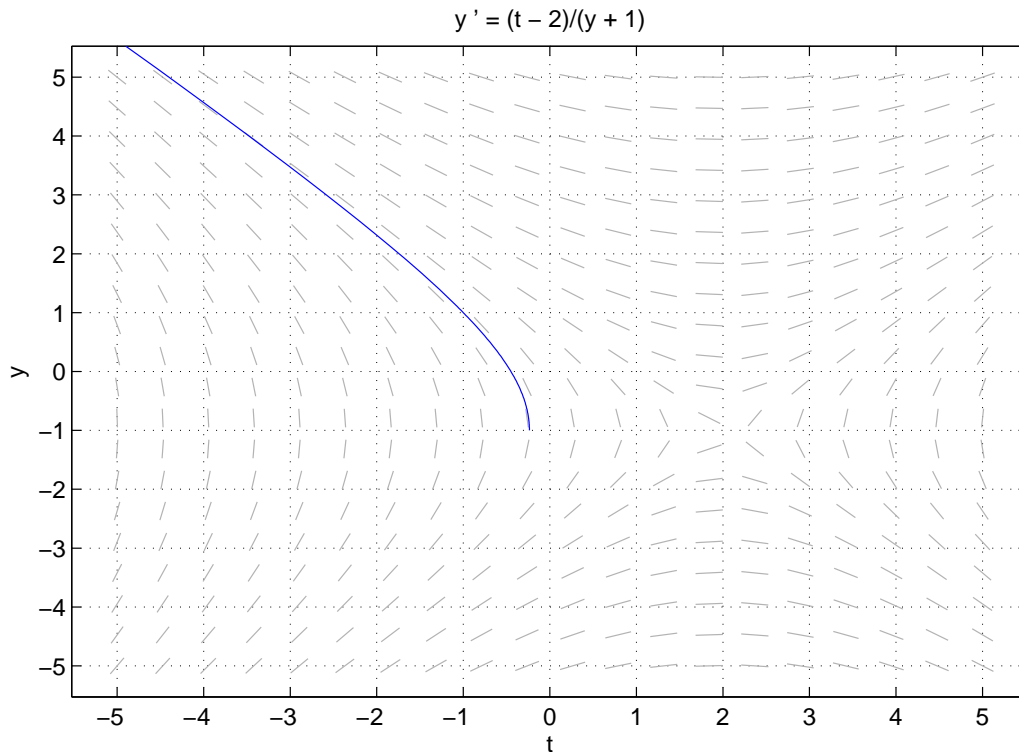


- There is no solution which satisfies the initial condition because if $t = 0$ then $y(t) = t(\sin(t) + C) = 0$, too. This doesn't contradict the existence theorem because our equation was not in normal form! If we rearrange the equation to normal form we get $y' = y/t + t \cos(t)$, which is not continuous at any point with t -coordinate equal to 0. Hence, the theorem does not apply.

Problem 9 Consider the initial value problem $y' = 3y^{2/3}$, $y(0) = 0$. We are given the functions $y_1(t) = t^3$ and $y_2(t) = 0$. It is easy to see that y_2 is a solution to the initial value problem, because in this case, everything in sight is always zero. It is a simple check to see that y_1 is also a solution. This does not contradict the uniqueness theorem, because the partial derivative of the right hand side of our equation with respect to y is $2y^{-1/3}$. As this function is not continuous at the point $(0, 0)$, the theorem does not apply.

For the next two exercises, there are two parts. First, we are to use a numerical solver to sketch the solution to the initial value problem, discuss difficulties the solver may have and estimate from the sketch what the interval of existence for the solution is. Second, we are to solve the equation by hand and find the actual interval of existence for comparison.

Problem 12 The initial value problem is $y' = \frac{t-2}{y+1}$, $y(-1) = 1$. We use `dfield` to sketch the solution, and we get the following picture.



From the picture, it appears that the interval of existence for the solution is $(-\infty, -0.235)$. (I zoomed a bit.)

To solve the equation by hand, notice that it is separable. It can be rearranged to read

$$(y + 1)y' = t - 2$$

Now we integrate to find the implicit solution. (Use the definite integral to save work.)

$$\frac{y^2}{2} + y = \frac{t^2}{2} - 2t - 1.$$

To get an explicit solution, multiply through by 2 and then add 1 to each side to obtain

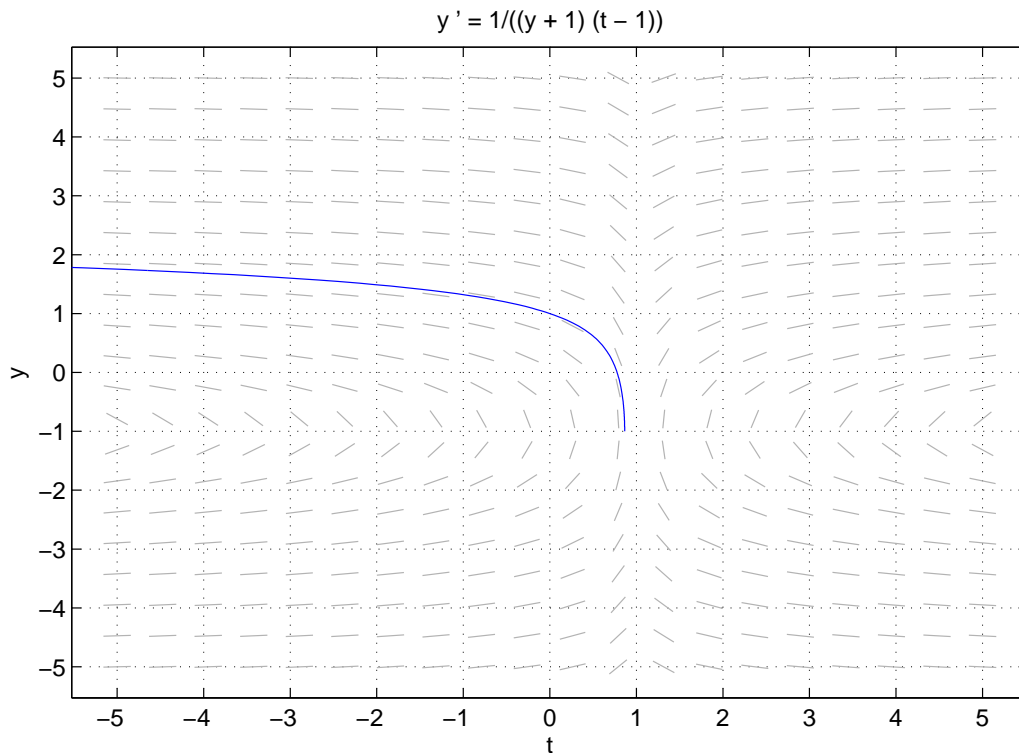
$$(y + 1)^2 = y^2 + 2y + 1 = t^2 - 4t - 1,$$

and hence our explicit solution is

$$y(t) = -1 + \sqrt{t^2 - 4t - 1}.$$

By checking when the term inside the square root is positive, we can see that the interval of existence is really $(-\infty, 2 - \sqrt{5})$. This compares well with our answer above, since $2 - \sqrt{5} \approx -0.2361$.

Problem 13 The initial value problem is $y' = \frac{1}{(t-1)(y+1)}$, $y(0) = 1$. We use `dfield` to sketch the solution, and we get the following picture.



From the picture, it appears that the interval of existence for the solution is $(-\infty, 0.86)$. (Again, I zoomed in.)

To solve the equation by hand, notice that it is separable. It can be rearranged to read

$$(y + 1)y' = \frac{1}{t - 1}$$

Now we integrate and rearrange to find the implicit solution.

$$\frac{y^2}{2} + y - \frac{3}{4} = \ln |t - 1|.$$

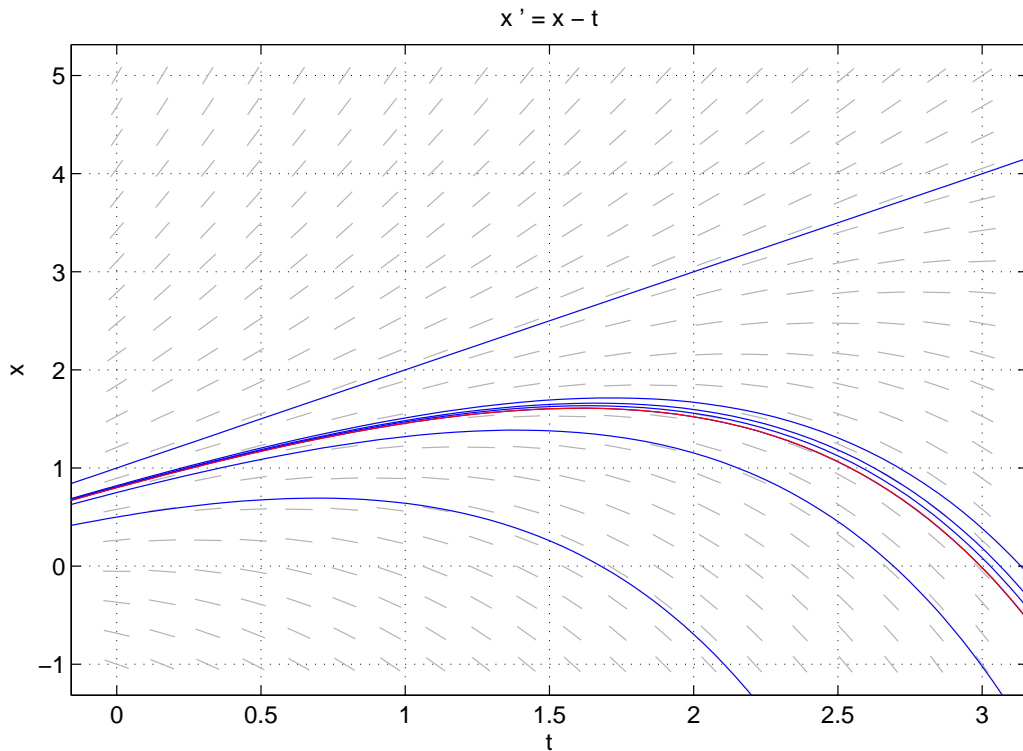
This requires the same type of trick as above to rearrange. We get an explicit solution of

$$y(t) = -1 + \sqrt{4 + 2 \ln |t - 1|}.$$

For this expression to make sense, we need to avoid $t = 1$ and we need the expression inside the square root to be positive. This works out to give the restriction that $t < 1 - e^{-2}$. Hence, the interval of existence is really $(-\infty, 1 - e^{-2})$. This compares well with our answer above, because $1 - e^{-2} \approx 0.8647$.

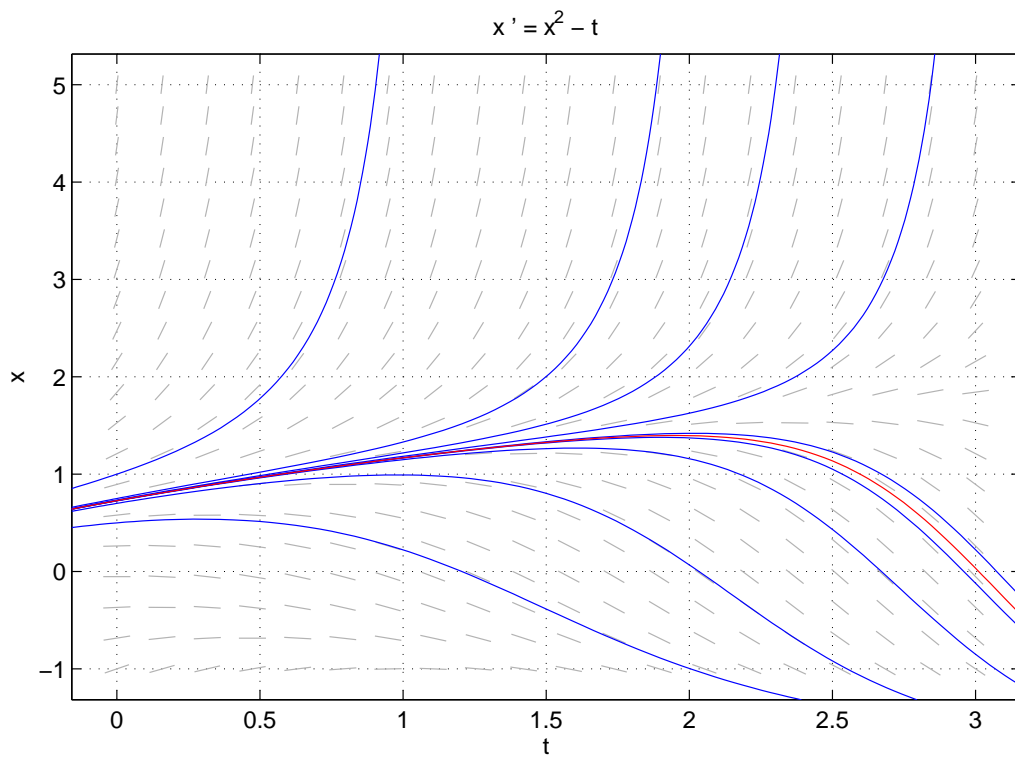
1.2. **Chapter 2.8.** For all of the exercises in this section, we are try some target practice with the `dfield` program. Our initial condition always happens at $t_0 = 0$, and we try to hit the specified point $(3, 0)$.

Problem 1 The equation is $x' = x - t$. I'll include a picture of my trials below.



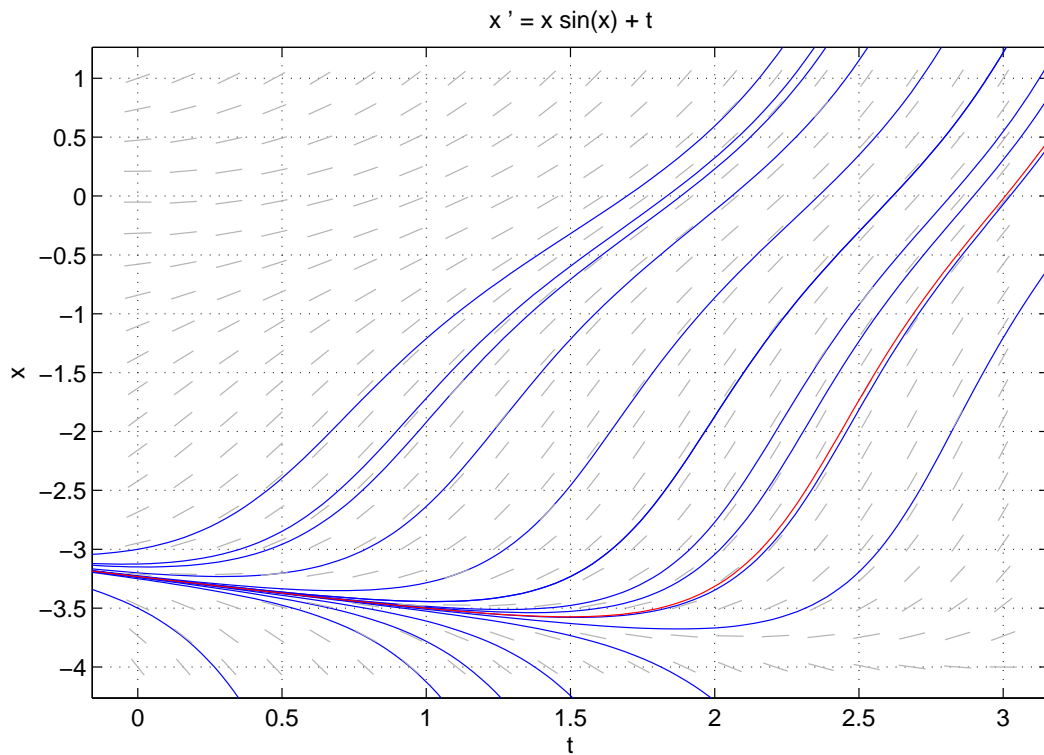
Eventually, I found that the starting position needed was $x = .8000$. The good curve is in red. I wrote all of those zeros because I'm fairly sure they are there. That is, I think they are significant digits.

Problem 5 The equation is $x' = x^2 - t$. I'll include a picture of my trials below.



Eventually, I found that the starting position needed was about $x = 0.72775$. The good curve is in red.

Problem 9 The equation is $x' = x \sin(x) + t$. I'll include a picture of my trials below.



Eventually, I found that the starting position needed was about $x = -3.231725$. The good curve is in red.

2. EXERCISES FROM THE MANUAL

2.1. Chapter 4.

Problem 10 First we solve the initial value problem $y' - y/t = -t \sin(t)$, $y(\pi) = -\pi$ by hand. This is a linear equation, I solved it with the integrating factor $u = 1/t$:

$$\left(\frac{1}{t}y\right)' = \frac{1}{t}y' - \frac{1}{t^2}y = \frac{1}{t}(y' - y/t) = \frac{1}{t}(-t \sin(t)) = -\sin(t).$$

So we integrate to get $y/t = \cos(t) + C$, or, equivalently, $y(t) = t \cos(t) + Ct$. To get the initial value correct, we must choose $C = 0$, so our solution is $y(t) = t \cos(t)$.

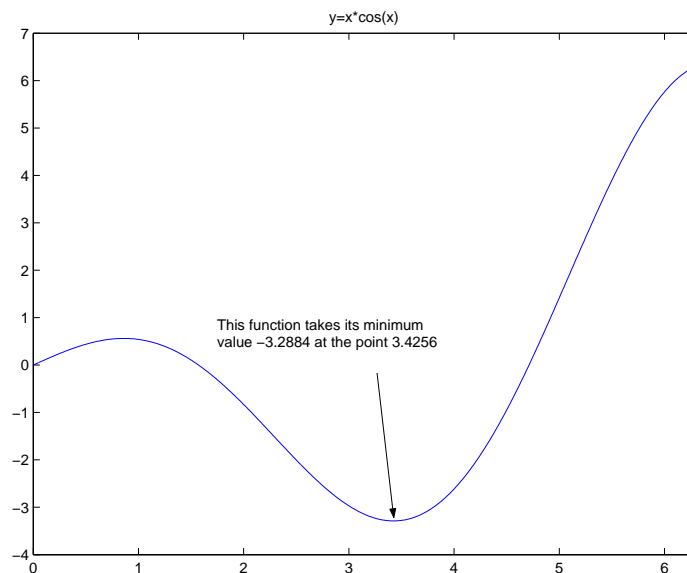
As for the MATLAB portion of the problem, the function m-file I wrote was:

```
function y = prob10(x)
```

```
% This is a MATLAB file for a function required in a  
% homework problem. See manual ch4, prob 10
```

```
y = x.*cos(x);
```

To get the minimum and where it occurs, you use the command `fminbnd` as follows. `[t,y]= fminbnd(@prob10,0,2*pi)`. This returns `t =3.4256`, `y= -3.2884`. The first output is the minima value and the second is the minimum value. We include our pretty picture below.



Problem 14 This problem is exactly like the last one, except we are given the initial value problem $y' = y \cos(t)$, $y(0) = 1$. This equation is separable, so we may solve it easily. The general solution is $y(t) = Ce^{\sin(t)}$. To match the initial condition, we must take $C = 1$, hence our solution is $y(t) = e^{\sin(t)}$.

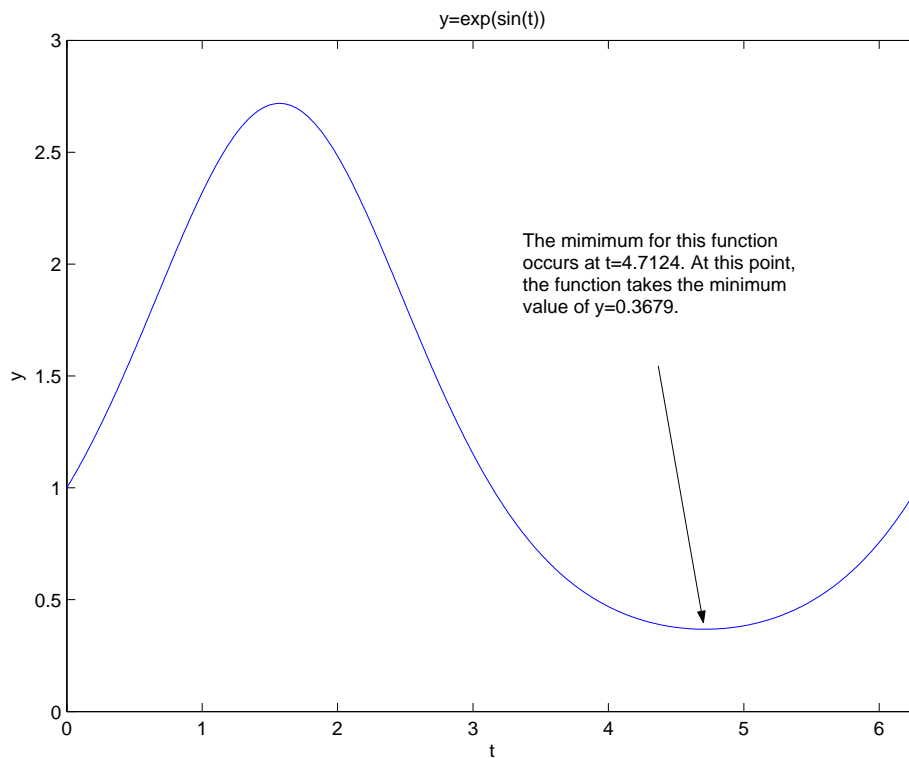
Again, we pass to the MATLAB portion of the problem. I wrote the function m-file below:

```
function y=prob14(x)
```

```
% This is a MATLAB file for a function required in a  
% homework problem. See manual ch4, prob 14
```

```
y=exp(sin(x));
```

We can then graph the function and use `fminbnd` just as above. The minima is at $t = 4.7124$ and the minimum value is $y = 0.3679$. Our pretty picture is included below.



Problem 20 For this problem we are to write some MATLAB code which allows us to easily graph a family of functions for several different values of the parameter. I wrote the following function m-file:

```
function y=prob20(x,C)

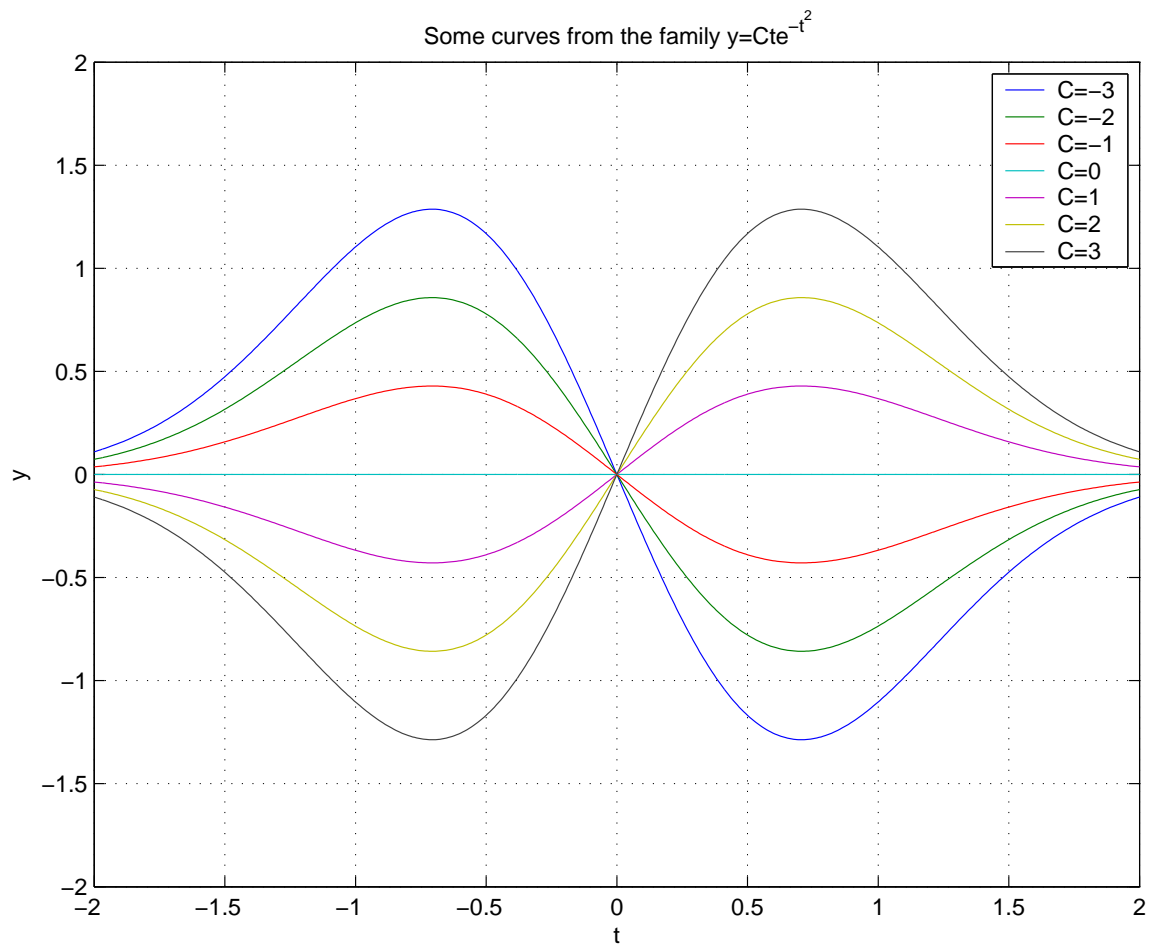
% This function implements a family of functions
% with parameter C which is required for prob 20
% in ch4 of the manual.

y=C*t.*exp(-t.^2);
```

And then I wrote a script m-file (which called the previous function) to make the graphs. This file was:

```
t=linspace(-2,2,500);
y=[];
for C=-3:3
    y=[y; prob20(t,C)];
end
plot(t,y)
xlabel('t')
ylabel('y')
title('Some curves from the family  $y=Cte^{-t^2}$ ')
axis([-2,2,0,2])
legend('C=-3','C=-2','C=-1','C=0','C=1','C=2','C=3')
shg
```

The resulting picture looks like this.



3. THE EXTRA EXERCISE

Problem This exercise has three parts related to the initial value problem

$$y' = y^2, \quad y(0) = 1.$$

- (1) Solve by hand. This equation is separable, so we need only rearrange $\frac{y'}{y^2} = 1$ and integrate to get

$$-\frac{1}{y(t)} - \left(-\frac{1}{1}\right) = \int_0^t \frac{y'(t)}{y(t)^2} dt = \int_0^t dt = t.$$

Or, in an explicit form, $y(t) = \frac{1}{1-t}$. This solution has interval of existence equal to $(-\infty, 1)$.

- (2) Use Picard iteration to find approximate solutions. Recall that we begin with the constant function

$$y(t) = y_0 = 1$$

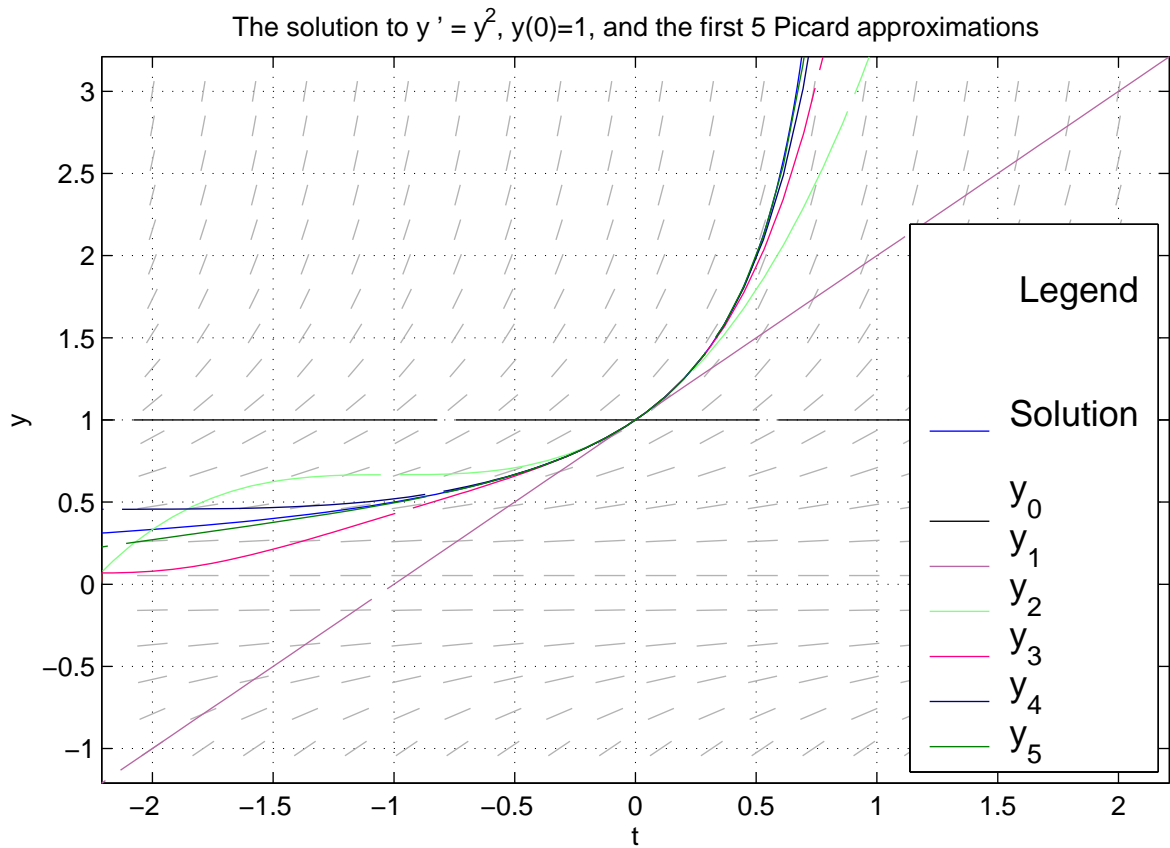
and continue by using the rule $y_{n+1}(t) = 1 + \int_0^t y_n(s)^2 ds$. This involves a fair amount of algebra, so I used MATLAB to compute symbolically. I'll attach a MATLAB diary to the end of the document. It can be difficult to see what is happening here—we have polynomials, but their coefficients are strange, and they have relatively high degree very quickly. But notice that after so many times, we get some thing like $y_5(t) = 1 + t + t^2 + \dots$. When we square such things and integrate, we will get *more* ones as coefficients! So, I conjecture that my approximates will eventually look like $1 + t + t^2 + \dots + t^n + \text{strange stuff}$. Another look at the weird high order terms can help. Notice that in most of the fractions, the denominator looks bigger than the numerator. If you use the `sym2poly` command in MATLAB to get a clear look at just the coefficients, you see that in both y_5 and y_6 , most of the coefficients of really high order are so close to zero that MATLAB can't tell the difference. And all of the coefficients are at most 1, with the ones for low order terms being spot on equal to 1. This also helps back up our conjecture. Of course, to be really sure, we should probably write a script to compute the first 300 or 400 terms and look at those.

Anyway, to see that this is going to the right place, we only need to recall that the power series for $\frac{1}{1-t}$ about 0 is

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + t^4 + t^5 + \dots$$

So everything makes sense. Notice that even though our actual solution has a restricted interval of existence, the approximate solutions given by Picard's method don't show this. They are all defined everywhere (since they are polynomials).

- (3) To make a picture on a direction field, I use `dfield` and the cut-and-paste of MATLAB to save work. My picture is below. (It is a bit ugly; I haven't mastered the graphing in `dfield` yet.)



Finally, here is the MATLAB diary I kept of my work for this problem.

```
t=sym('t'); y0=1; y1=1+int(y0^2,t,0,t)
```

```
y1 =
```

$$1+t$$

```
y2=1+int(y1^2,t,0,t)
```

```
y2 =
```

$$1+1/3*t^3+t^2+t$$

```
y3=1+int(y2^2,t,0,t); pretty(y3)
```

$$1 + \frac{1}{63}t^7 + \frac{1}{9}t^6 + \frac{1}{3}t^5 + \frac{2}{3}t^4 + t^3 + t^2 + t$$

```
y4=1+int(y3^2,t,0,t); pretty(y4)
```

$$1 + \frac{1}{59535}t^{15} + \frac{1}{3969}t^{14} + \frac{1}{567}t^{13} + \frac{1}{126}t^{12} + \frac{5}{189}t^{11} + \frac{22}{315}t^{10}$$

$$+ \frac{86}{567}t^9 + \frac{71}{252}t^8 + \frac{8}{63}t^7 + \frac{29}{63}t^6 + \frac{2}{3}t^5 + \frac{13}{15}t^4 + \frac{5}{15}t^3 + \frac{4}{15}t^2 + \frac{3}{15}t + \frac{2}{15}$$

```
y5=1+int(y4^2,t,0,t); pretty(y5)
```

$$1 + t^2 + t^3 + \frac{43}{45}t^6 + \frac{6}{45}t^5 + \frac{5}{45}t^4 + \frac{13}{15}t^7 + \frac{7}{15}t^6 + \frac{1019}{8820}t^{14} + \frac{14}{73710}t^{13} + \frac{13141}{68040}t^{12} + \frac{17779}{68040}t^{11}$$

$$+ \frac{1477}{4050}t^{11} + \frac{27523}{56700}t^{10} + \frac{10}{5670}t^9 + \frac{3497}{5670}t^8 + \frac{9}{1260}t^7 + \frac{943}{1260}t^6 + \frac{8}{893025}t^5 + \frac{63283}{893025}t^4 + \frac{15}{893025}t^3$$

$$+ \frac{1}{3544416225}t^{30} + \frac{1}{236294415}t^{29} + \frac{13}{315059220}t^{28} + \frac{2}{6751269}t^{27}$$

$$\begin{aligned}
& + \frac{1}{595350} t^{26} + \frac{5309}{675126900} t^{25} + \frac{16927}{540101520} t^{24} + \frac{2447}{22504230} t^{23} \\
& + \frac{557}{1666980} t^{22} + \frac{207509}{225042300} t^{21} + \frac{16511}{7144200} t^{20} + \frac{162179}{30541455} t^{19} \\
& + \frac{2588}{229635} t^{18} + \frac{1080013}{48580560} t^{17} + \frac{43363}{1058400} t^{16} + \frac{1}{109876902975} t^{31}
\end{aligned}$$

sym2poly(y4)

ans =

Columns 1 through 7

0.0000 0.0003 0.0018 0.0079 0.0265 0.0698 0.1517

Columns 8 through 14

0.2817 0.4603 0.6667 0.8667 1.0000 1.0000 1.0000

Columns 15 through 16

1.0000 1.0000

sym2poly(y5)

ans =

Columns 1 through 7

0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000

Columns 8 through 14

0.0000 0.0001 0.0003 0.0009 0.0023 0.0053 0.0113

Columns 15 through 21

0.0222 0.0410 0.0709 0.1155 0.1783 0.2613 0.3647

Columns 22 through 28

0.4854 0.6168 0.7484 0.8667 0.9556 1.0000 1.0000

Columns 29 through 32

1.0000 1.0000 1.0000 1.0000

'And just for good measure'

ans =

And just for good measure

y6=1+int(y5^2,t,0,t); sym2poly(y6)

ans =

Columns 1 through 7

0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000

Columns 8 through 14

0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000

Columns 15 through 21

0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000

Columns 22 through 28

0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000

Columns 29 through 35

0.0001 0.0002 0.0004 0.0007 0.0012 0.0020 0.0034

Columns 36 through 42

0.0057	0.0090	0.0141	0.0215	0.0319	0.0463	0.0657
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Columns 43 through 49

0.0911	0.1235	0.1640	0.2133	0.2718	0.3393	0.4150
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Columns 50 through 56

0.4976	0.5846	0.6729	0.7587	0.8375	0.9046	0.9556
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Columns 57 through 63

0.9873	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
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Column 64

1.0000

diary off