HOMEWORK # 5 SOLUTIONS

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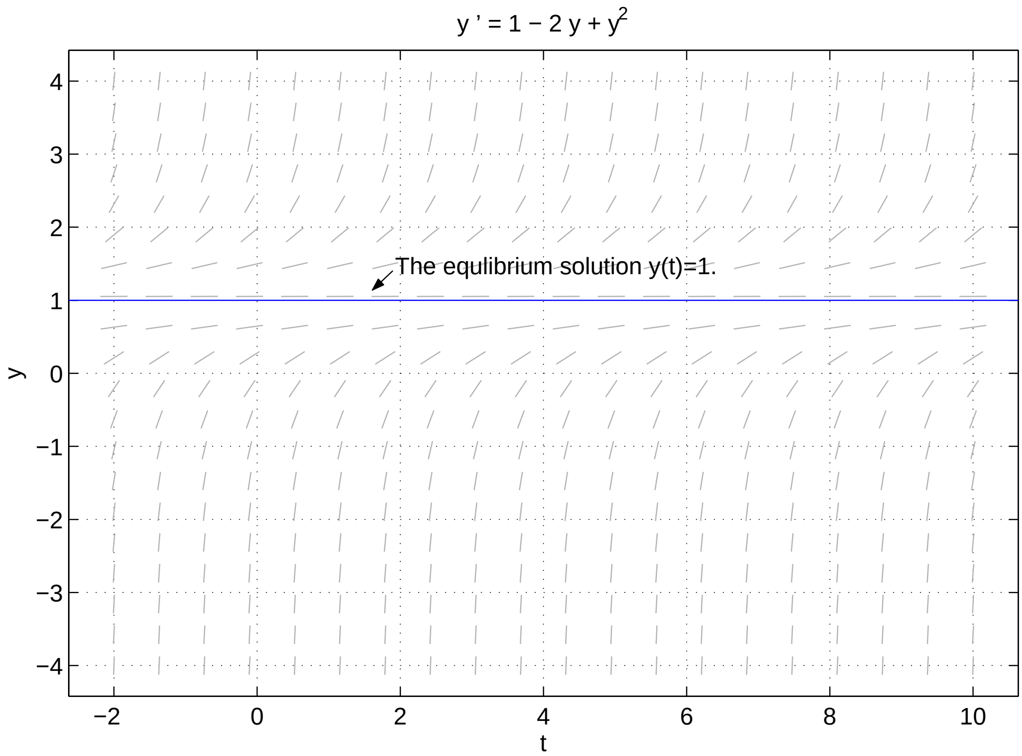
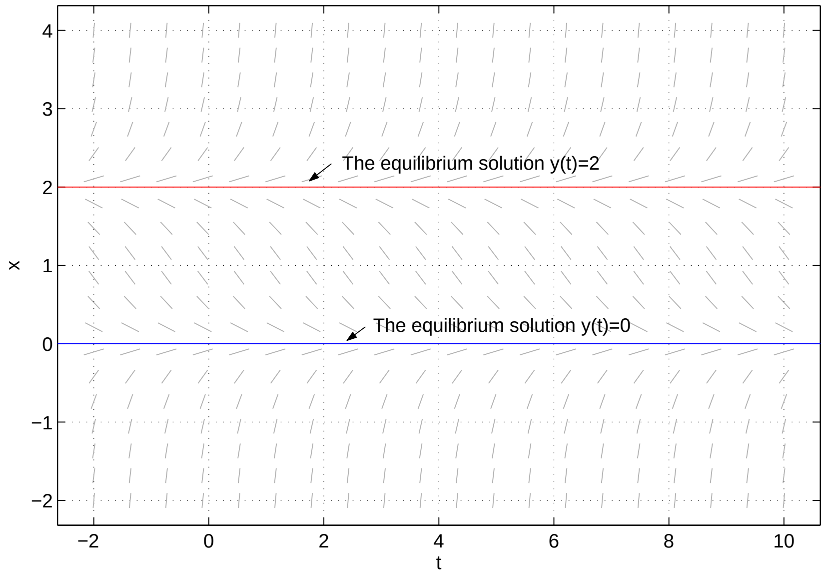
1. Exercises from the text

1.1. Chapter 2.9.

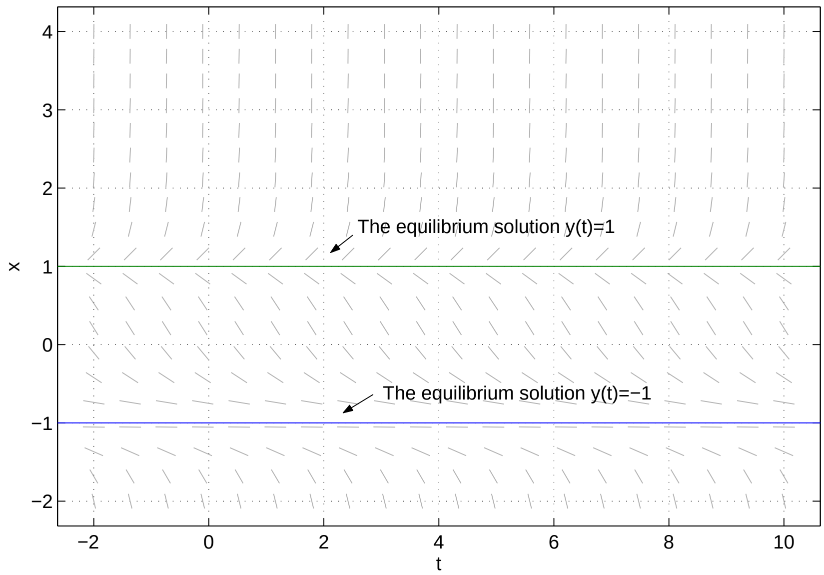
Problem 2 We are to consider the equation $y' = 1 - 2y + y^2$. The equation is autonomous, and by looking for roots of the right hand side, we find that the only equilibrium solution is $y(t) = 1$. It is not difficult to see that this is an unstable solution. Technically, we could call this a half stable solution, as solutions below $y = 1$ tend to increase up to 1, and solutions above $y = 1$ also increase, but this carries them away from $y = 1$. We plot a direction field with the equilibrium solution on it below.



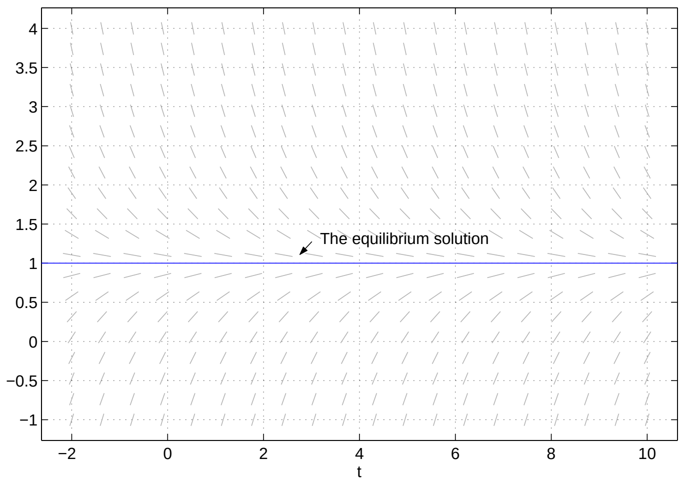
Problem 8 From the given graph of $f(y)$, it is clear that there are two equilibrium solutions to $y' = f(y)$. One of them, $y(t) = 0$ is asymptotically stable, and the other, $y(t) = 2$, is unstable. We plot the equilibrium solutions in the ty-plane below.



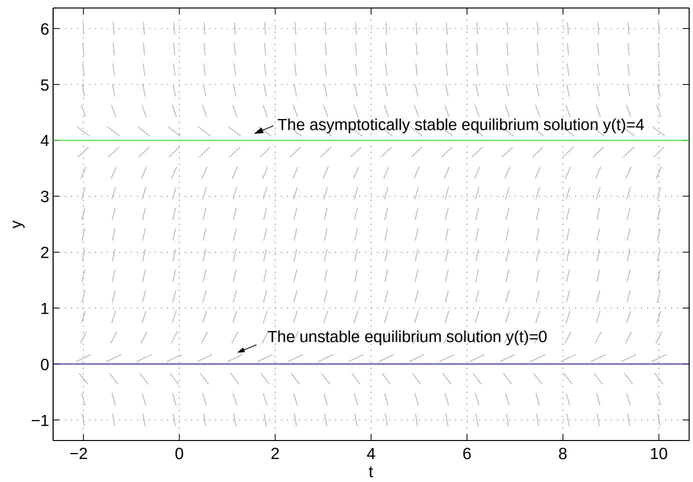
Problem 9 From the given graph of $f(y)$, it is clear that there are two equilibrium solutions to $y' = f(y)$. One of them, $y(t) = 1$ is unstable, and the other, $y(t) = -1$, is half stable. The text refers to both as unstable. We plot these equilibrium solutions in the ty-plane below.



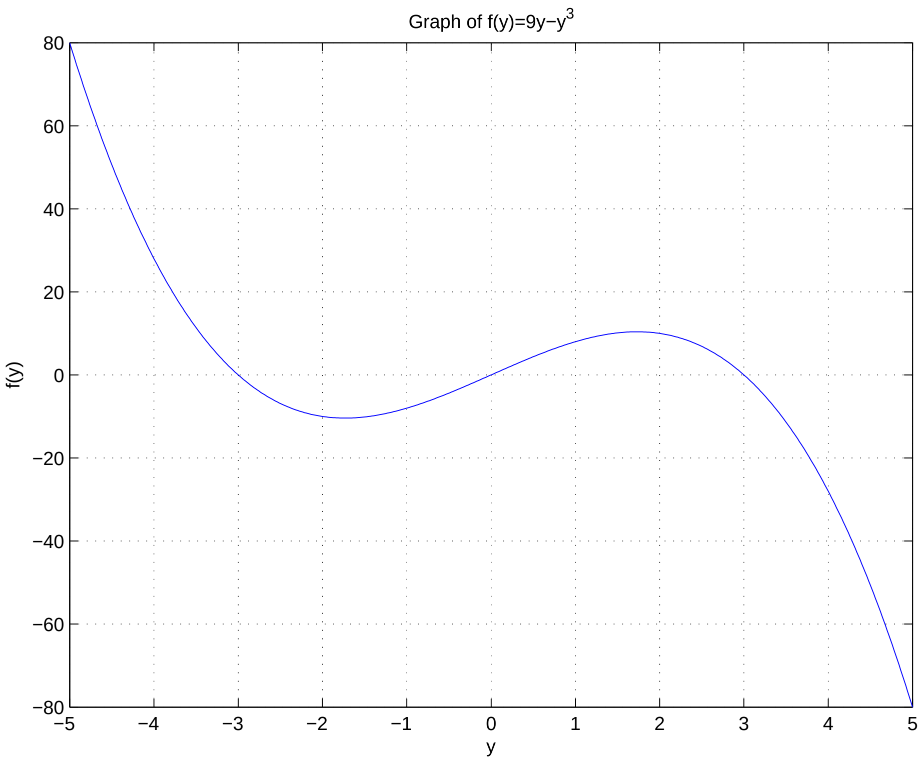
Problem 11 As the equation is autonomous, we need only to continue the sketch of the direction field by shifting the given strip to the right and left until it fills up our rectangle. In this problem, there is only one equilibrium solution to be found; it is located at the spot where the direction field is horizontal. It is clear from the rest of the picture that this is an asymptotically stable equilibrium. To make a picture for you, I picked a function which had the correct behavior and used `dfield` (which is technically cheating).

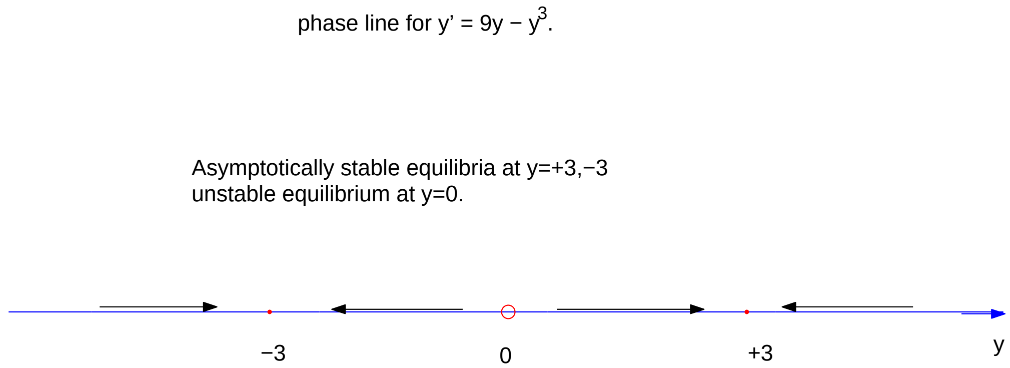


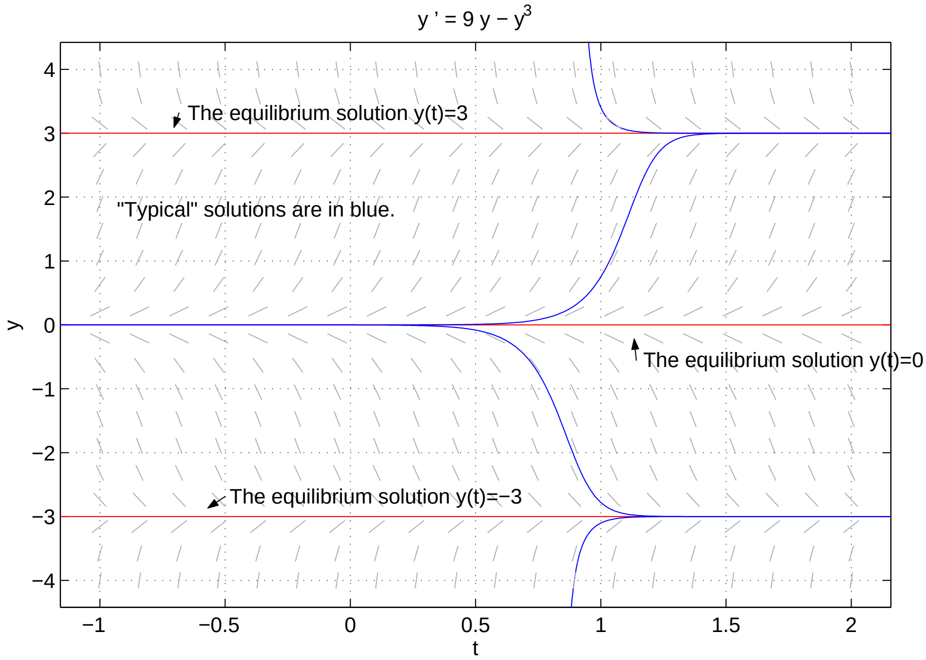
Problem 14 From the graph of $f(y)$, it is clear that there are two equilibria. A stable one at $y(t) = 4$ and an unstable one at $y(t) = 0$. Again, I cheated and picked a function which represents the correct behavior to get a graph instead of sketching by hand.



Problem 19 Again, I will not sketch by hand, but use MATLAB to give the correct graphs for the solution set. You should do this to check your answers if you are not sure, anyway. Our equation is $y' = 9y - y^3 = y(9 - y^2) = y(3 + y)(3 - y)$. We give a sketch below. From the sketch, it is clear that there are three equilibrium points. Two asymptotically stable ones at $y = -3$ and $y = 3$ with an unstable one in between at $y = 0$. We give a picture of a phase line, too. Finally, we give a picture of the equilibrium solutions and some representative solutions from the regions in between on a direction field.







Problem 25 Consider the initial value problem $y' = (1 + y)(5 - y), \quad y(0) = 2$. We can solve this problem analytically by separating variables and using a partial fraction decomposition. Cobbling a lot of steps together, we get

$$t = \int_0^t ds = \int_2^y \frac{dw}{(1+w)(5-w)} = \int_2^y \frac{1}{6}\left(\frac{1}{1+w} + \frac{1}{5-w}\right) dw$$
$$= \frac{1}{6}(\ln|1+w| - \ln|5-y|) - \frac{1}{6}(\ln(1) - \ln(5))$$

This needs some rearranging to get $6t = \ln\left|\frac{5(1+y)}{5-y}\right|$, or $\frac{5(1+y)}{5-y} = e^{6t}$. (This choice of sign for the absolute value is consistent with the initial condition $y(0) = 2$. Finally, we do some algebra to obtain an explicit expression.

$$y(t) = \frac{e^{6t} - 1}{e^{6t}/5 + 1} = 5\left(\frac{e^{6t} - 1}{e^{6t} + 5}\right).$$

As t grows and limits on $+\infty$, we see that $y(t) \to 5$. This was some fairly hard work.

For comparison, we use a phase line analysis. From the form of the equation, we see we have two equilibrium points. One at $y = -1$ and one at $y = 5$. Our defining function is positive on the interval $(-1, 5)$, so any solution in this region must increase and eventually limit on $y(t) = 5$ as $t \to +\infty$. This way is much easier.

Problem 27 In this problem we are to use the first derivative test for stability to determine the nature of the equilibrium points in the equation $x' = f(x) = 4 - x^4$. The zeros of f are at $\pm\sqrt{2}$, so these are our equilibrium points. The derivative of f is $f'(x) = -4x^3$. It is easy to check that $f'(\sqrt{2}) < 0$ so that $\sqrt{2}$ is an asymptotically stable equilibrium, and that $f'(-\sqrt{2}) > 0$ so that $-\sqrt{2}$ is an unstable equilibrium.

1.2. **Chapter 3.1.**

Problem 5 We are given a Malthusian model of population growth with reproductive rate r, which is modified to have harvesting, too, with a constant rate of h organisms per hour. We are to analyze the fate of the culture using qualitative analysis. Introducing harvesting as above changes the differential equation to read

$$\frac{dP}{dt} = rP - h$$

where $P(t)$ is the population at time t. This equation has an equilibrium solution of $P(t) = h/r$, and this equilibrium is asymptotically stable. Therefore, we conclude that in the long run, the population will settle down to the value of h/r organisms. Notice that for this to make sense, we need to have both r and h positive.

Problem 15 This problem is similar to the last one. We are given a population of fish governed by the logistic equation $P' = 0.1P(1 - P/10)$, where P is thousands of fish, and time is measured in days. We then suppose that fishing begins at the rate of 100 fish per day. This adjusts the model to read as follows.

$$P' = 0.1P(1 - P/10) - 0.1 = (P(1 - P/10) - 1)/10.$$

We need to find the roots of $f(P) = (P(1-P/10)-1)/10 = -P^2/100 + P/10 - 1/10$. They are $P = 5 \pm \sqrt{15}$. We check the stability of these with the first derivative test. We find that $f'(P) = -P/50 + 1/10$. It is then simple to check that $5 - \sqrt{15}$ is an unstable equilibrium and $5 + \sqrt{15}$ is an asymptotically stable equilibrium.

This tells us that the eventual population of the fish in the lake should be approximately $P = 8.873$ thousand fish, as long as we begin above the other equilibrium of $P = 1.127$ thousand fish.

In particular, if you begin with 2000 fish, you will eventually have $8,873$ fish. However, if you start with $1,000$ fish, you are forced down away from the unstable equilibrium, hence the fish will die in a finite (and relatively short) amount of time.