

HOMEWORK # 6 SOLUTIONS

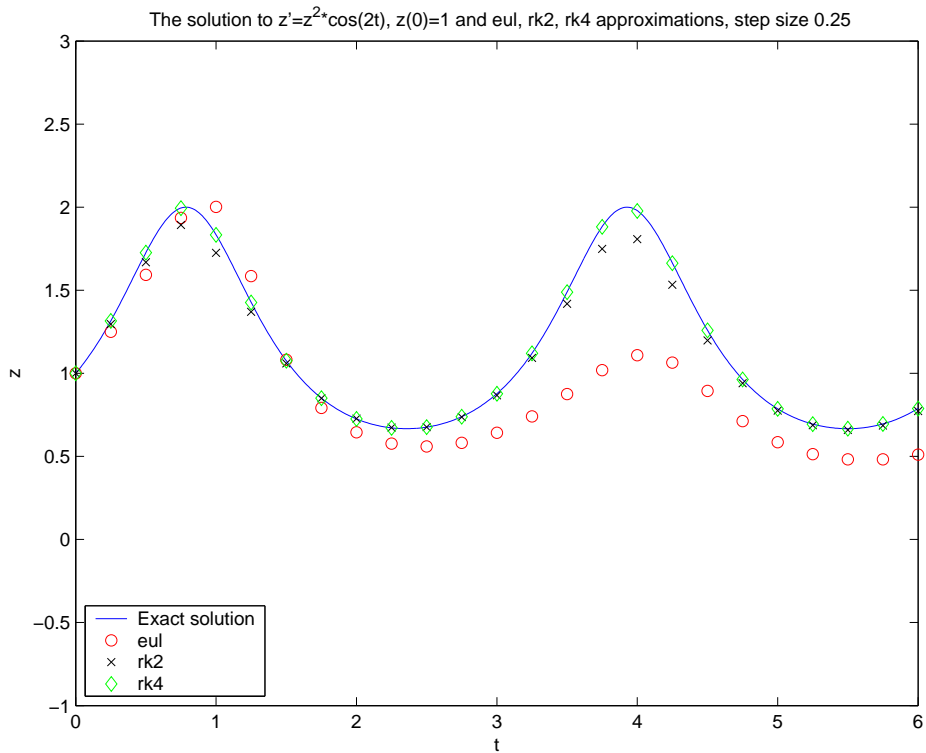
TJ HITCHMAN

1. EXERCISES FROM THE MANUAL CHAPTER 5

Problem 3. First, we work out the exact solution to the initial value problem $z' = z^2 \cos(2t)$, $z(0) = 1$. Since the equation is separable, this is not too difficult. We obtain

$$z = \frac{2}{2 - \sin(2t)}.$$

Now we use MATLAB to try our three basic numerical methods on this problem and plot the outputs. We are concerned with the interval $0 \leq t \leq 6$ and use step size $h = 0.25$. My plot is below.

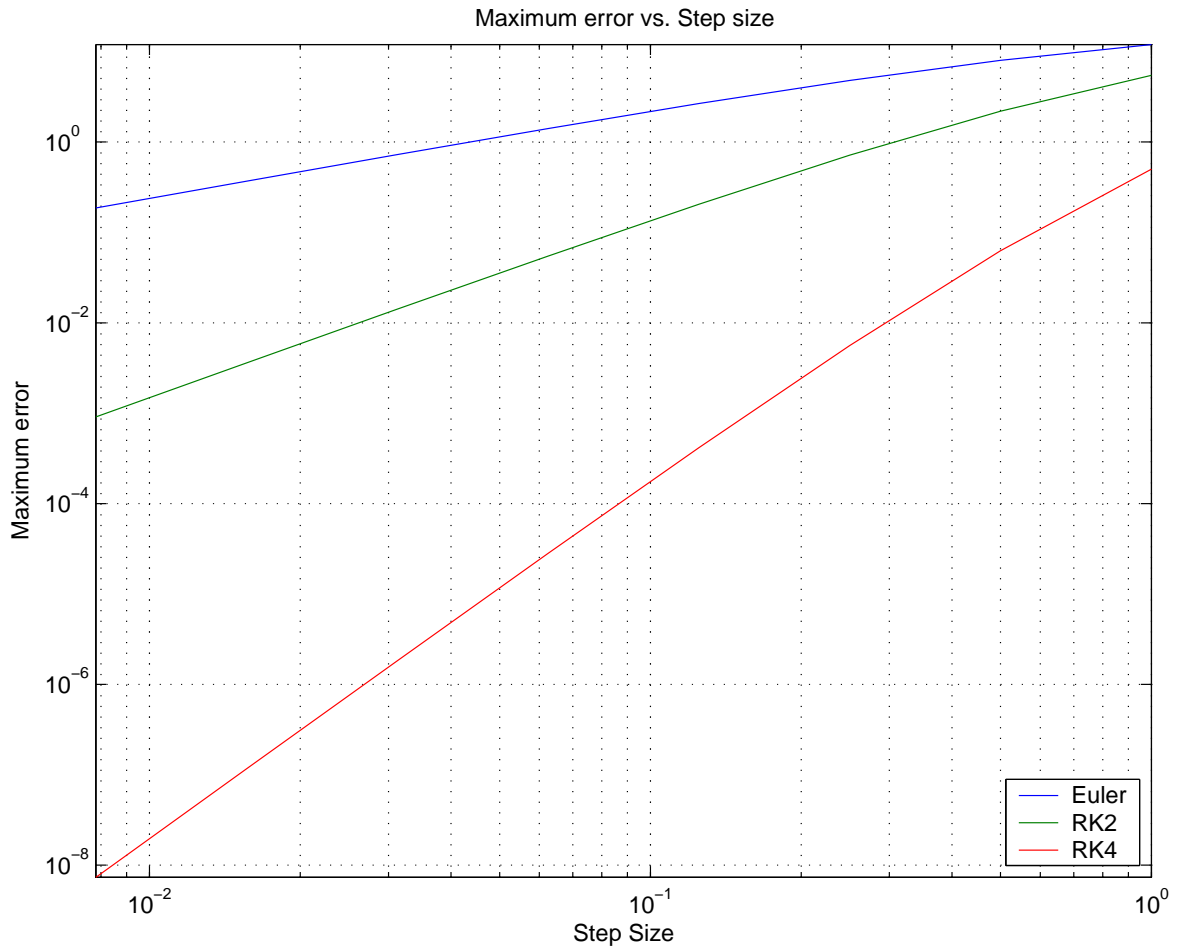


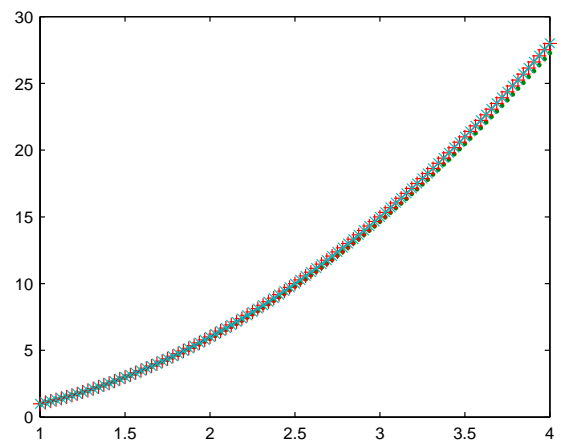
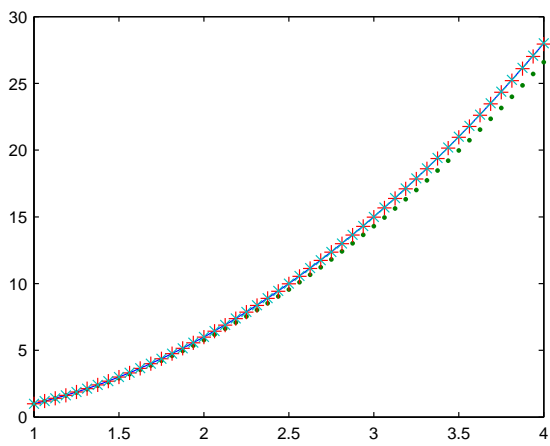
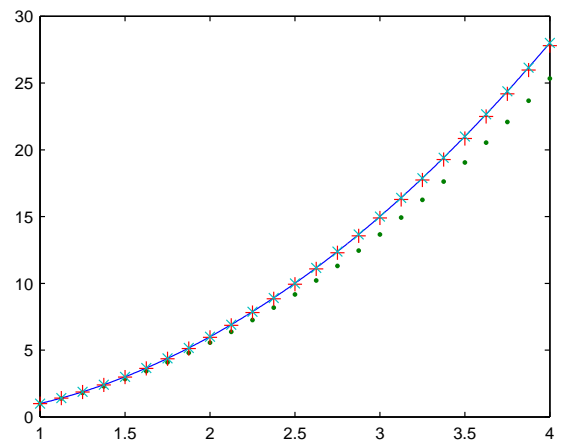
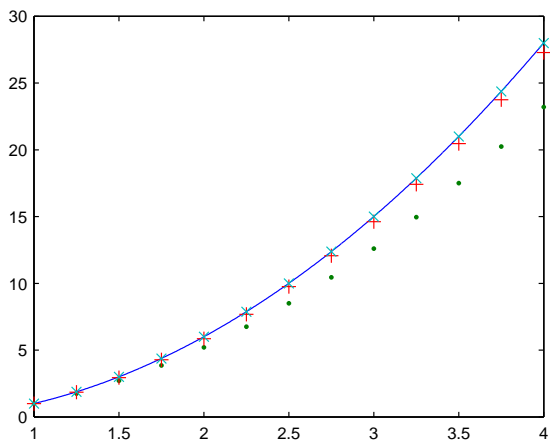
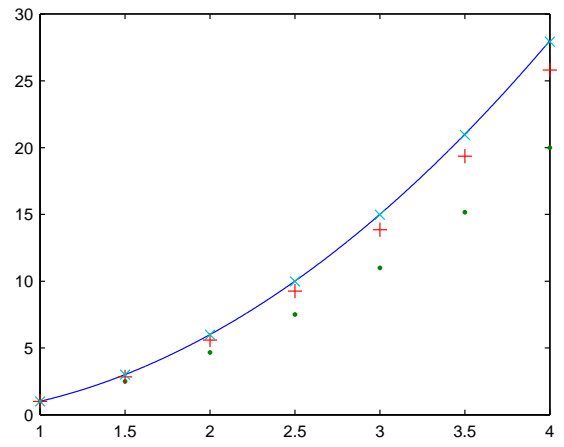
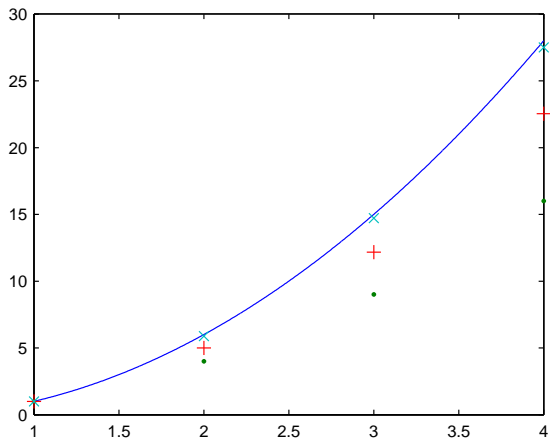
In the next two problems, we adapt the example from the text to study the error in our approximation routines for the given initial value problems on the given intervals. To combine the graphs, I used the `multigraf` `m`-file available on the web.

Problem 8. We are to consider the initial value problem $y' = 1 + 2y/t$, $y(1) = 1$ on the interval $[1, 4]$. The exact solution is $y = 2t^2 - t$.

The first six iterations of output of the code are on the following page. I have decided to drop the last two because the overlaps get worse and the pictures are even muddier. The graphs shown have step sizes of 1, 1/2, 1/4, 1/8, 1/16 and 1/32. In each, the exact solution is a blue line, the Euler method is given by green dots, the RK2 method by red pluses and the RK4 method by cyan crosses.

The final comparison of maximum errors are shown in the figure below on a log-log scale.

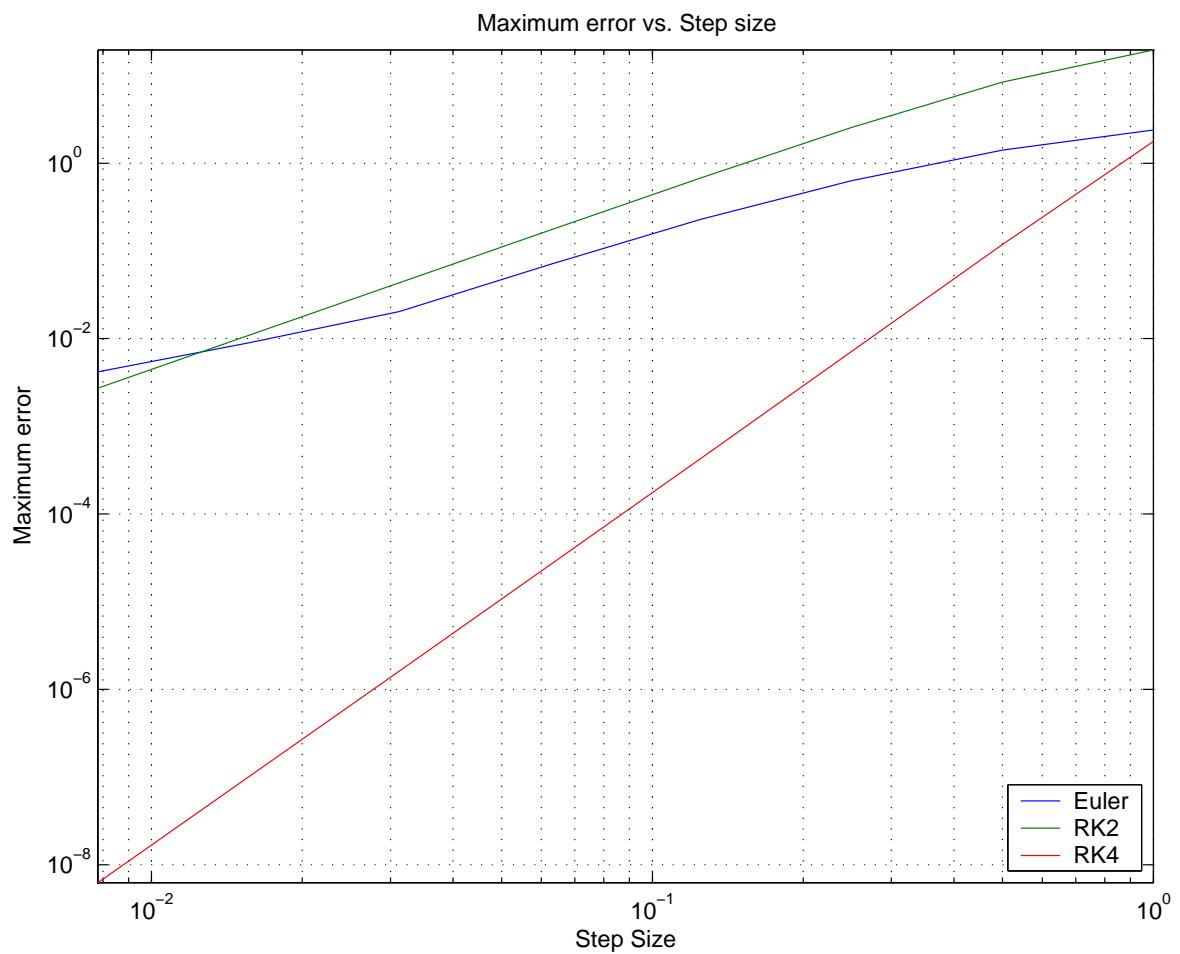


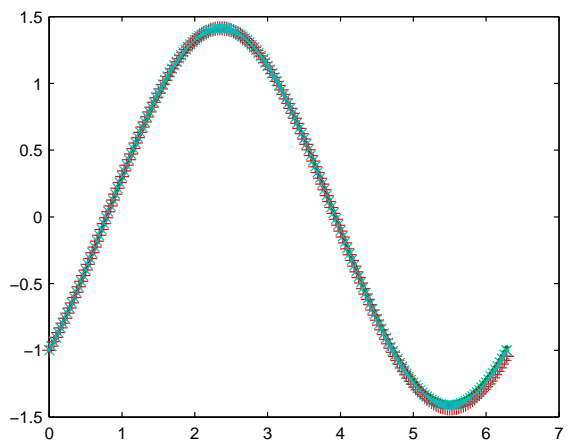
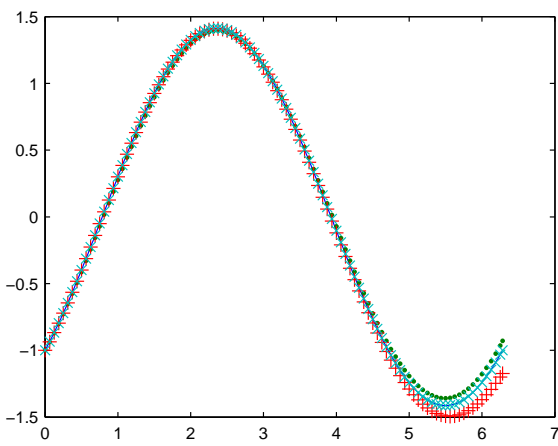
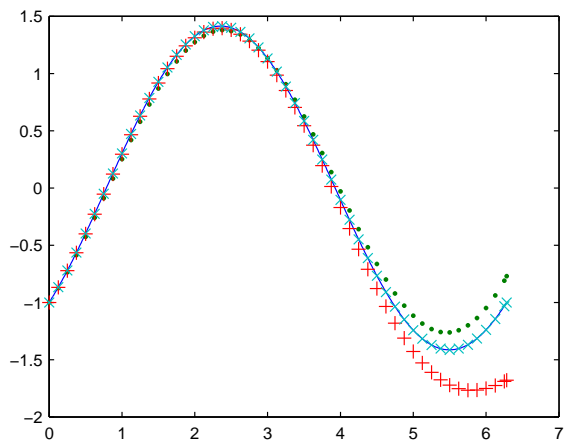
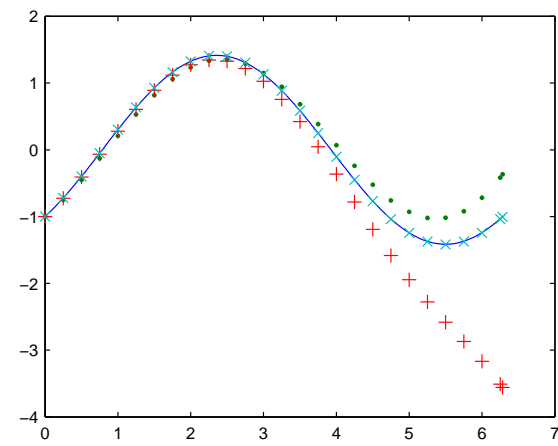
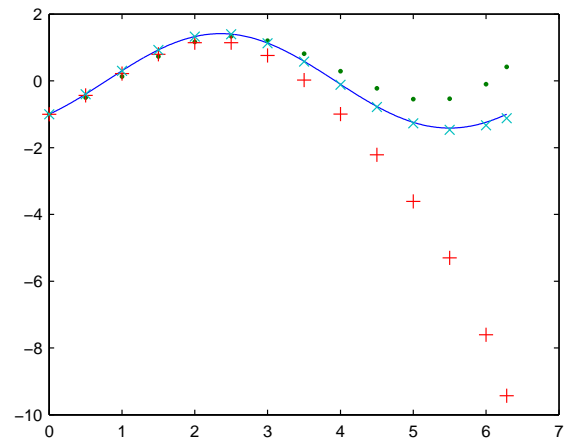
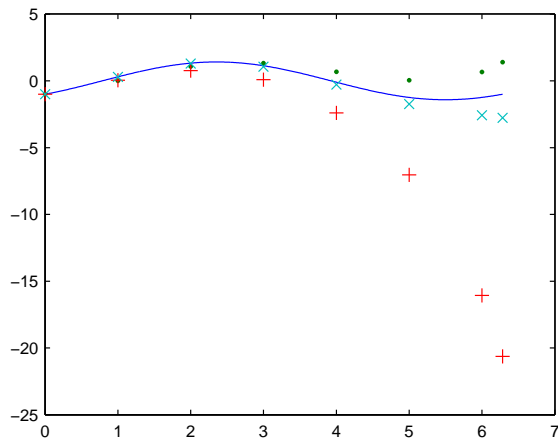


Problem 11. We are to consider the initial value problem $y' = y + 2 \cos(t)$, $y(0) = -1$ on the interval $[0, 2\pi]$. The exact solution is $\sin(t) - \cos(t)$.

The outputs of the code are on the next page. Again, I have dropped the last two and the graphs shown have step sizes of 1, 1/2, 1/4, 1/8, 1/16 and 1/32. Notice that the last step is always short. Our step sizes do not divide the interval up evenly (because it has length 2π), so there is a short "left over" step at the end.

Our log-log comparison of the errors is given in the graph below.





Problem 22. In this problem we are to investigate the usefulness of our methods on the stiff system $x' = e^t \cos(x)$, $x(0) = 1$. I plotted the graphs asked for and will include them on the following pages. It seems that eventually things break as t grows (and hence the derivative gets larger), and the solution curve oscillates wildly, even crossing the solution $x = \pi/2$. As this violates the uniqueness theorem, we can be sure our solvers are having difficulty. I also halved the step size two more times and made plots, but things got no better.

Problem 23. In this problem we investigate the accuracy of our numerical methods on two initial value problems which have the same solution. I wrote a bunch of m-files: two function files to define the differential equations and a script file to run the solvers and test for error sizes.

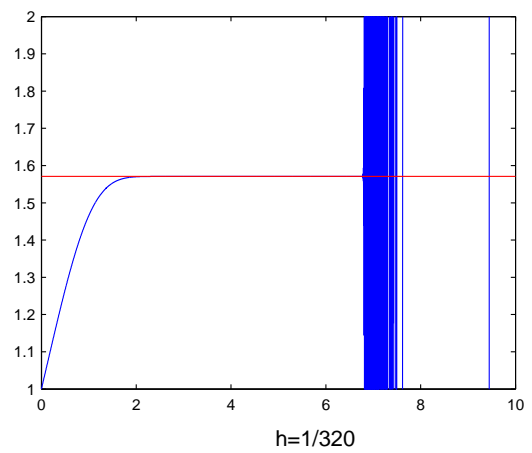
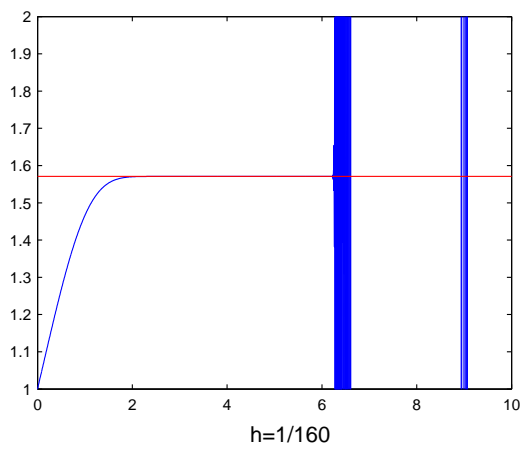
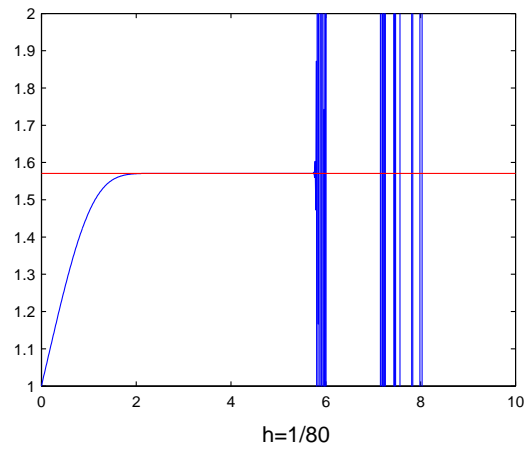
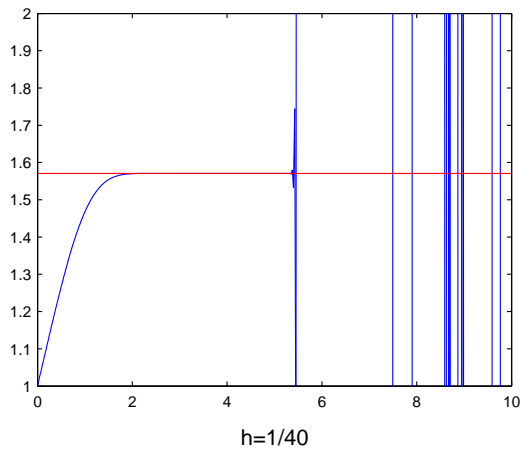
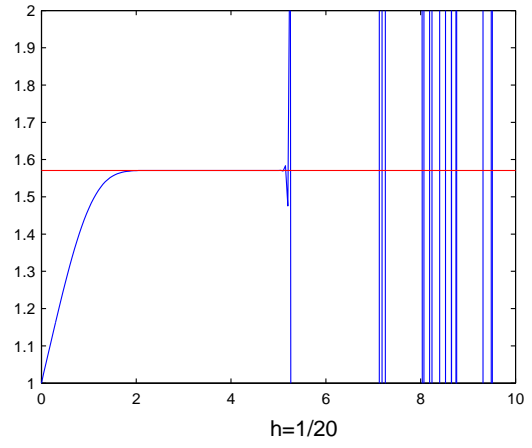
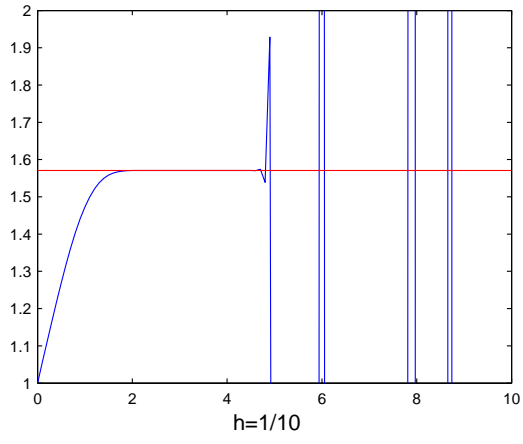
The maximum errors I got are tabulated below. The final column in the table is a comparison of the two errors for a given method. It shows how big the error in $y' = e^t$ is as a percentage of the error in $y' = y$.

Method	$y' = y$	$y' = e^t$	comparison
Euler	0.0135	0.0086	36.32%
RK2	4.4966e-005	1.4319e-005	68.16%
RK4	2.2464e-010	5.9655e-012	97.34%

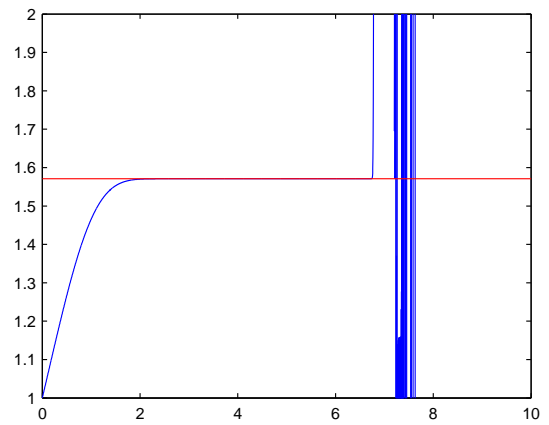
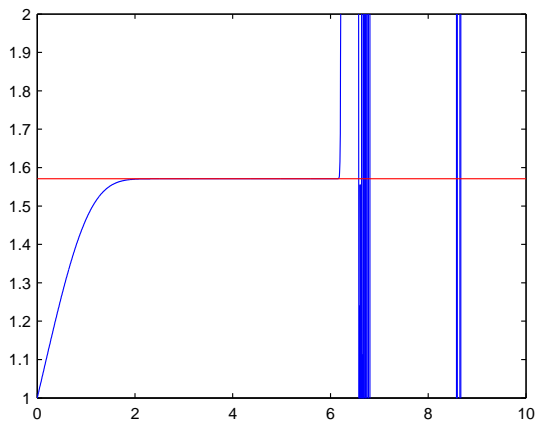
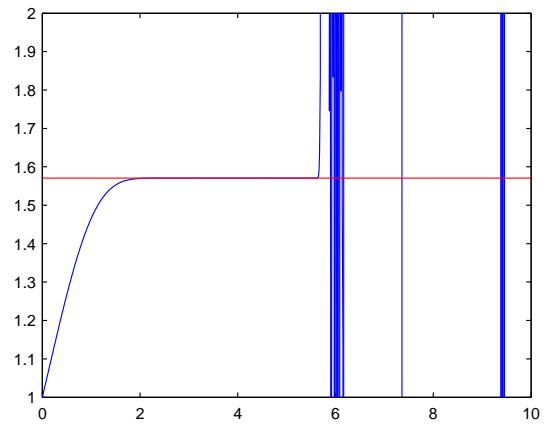
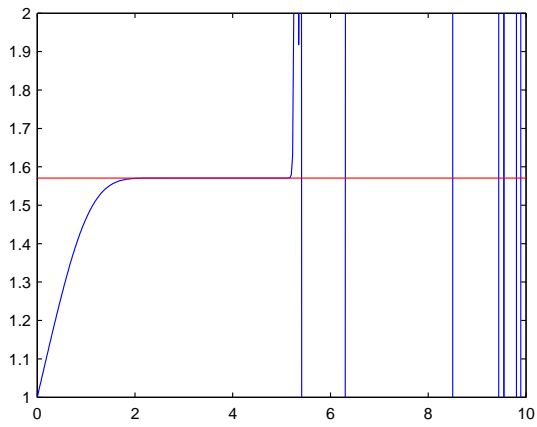
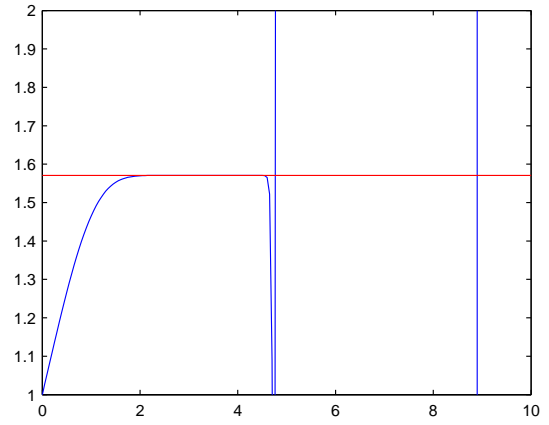
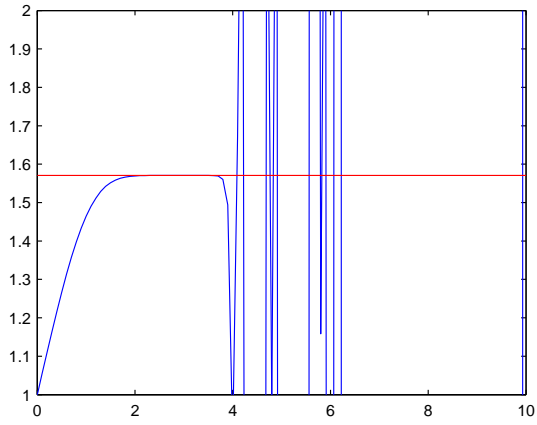
One should note that the errors are larger for the problem $y' = y$ than they are for $y' = e^t$, and also that this effect seems to be lessened in the higher order methods.

Graphs for problem 22 follow. All step sizes correspond.

Euler's method



Runge-Kutta order 2



Runge-Kutta order 4

