HOMEWORK # 8 SOLUTIONS

1. Exercises from the Text Chapter 7

1.1. **Section 7.6.** I'm just going to talk through what happens here. Many of these "by hand" computations are a pain to type out.

Problem 7 To put this into row echelon form, we only need to perform three operations: Add 6 times row one to row three, add -1 times row one to row two and then add three times row two to row one. At the end of this the matrix is upper triangular, and the bottom row is all zeros. Hence the determinant is zero.

Problem 11 Again, I performed the following operations in order: add 1/2 of row one to row three, add half of row one to row four, add -3/4 times row two to row three, add -5/4 times row two to row four, add -5/3 times row three to row four. At this stage, the matrix is upper triangular with product of the diagonal elements equal to $2 \cdot 2 \cdot 3 \cdot (2/3) = 8$. Hence the determinant of this matrix is 8.

Problem 16 Now we expand the matrix in exercise 7 along the top row. This yields that the determinant is equal to

$$3 \cdot \det \begin{pmatrix} 6 & 3 \\ -18 & -9 \end{pmatrix} - 0 + 0 = 3(6 \cdot (-9) - (-18) \cdot 3) = 0.$$

Problem 20 In this problem we compute the determinant of the matrix of exercise 11 by expanding along the third row. We see that the determinant is equal to

$$(-1) \det \begin{pmatrix} -1 & 3 & 4 \\ 2 & -2 & 0 \\ 3 & 1 & 2 \end{pmatrix} - 2 \det \begin{pmatrix} 2 & 3 & 4 \\ 0 & -2 & 0 \\ -1 & 1 & 2 \end{pmatrix} + 0 - 0$$

We expand both of these determinants along their second rows to get

$$(-1) \left[-(2) \det \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} + (-2) \det \begin{pmatrix} -1 & 4 \\ 3 & 2 \end{pmatrix} - 0 \right] - 2 \left[-0 + (-2) \det \begin{pmatrix} 2 & 4 \\ -1 & 2 \end{pmatrix} + 0 \right]$$

Which is equal to

$$(-1)(-2 \cdot (2) - 2(-14)) + 4(8) = -24 + 32 = 8$$

Problem 27 By putting this matrix in reduced row echelon form, we get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus the determinant must be zero, the nullspace is non-trivial, one dimensional and spanned by the vector $(0, -2, 1)^T$, and the columns are a linearly dependent set. One computation, three answers.

Problem 48 This *looks* hard. I even hinted to use MATLAB because the matrix is 6 by 6 with three digit numbers in it. A closer look reveals that the second and fourth rows are identical, hence the determinant must be zero, and hence the nullspace must be non-trivial. Note that the problem doesn't actually ask for a nonzero element of the nullspace explicitly.

2. EXERCISES FROM THE TEXT CHAPTER 8

2.1. **Section 8.1. Problem 2** This is a two dimensional autonomous system.

Problem 6 This is a four dimensional autonomous system.

Problem 7 This is a direct check. The point being is that things are just as they were for a single equation, all you have to do is check that your equations are satisfied by the given functions.

Problem 12 We introduce a new variable $v = x'$. Then the system can be written in vector notation as

$$\begin{pmatrix} x \\ v \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -k/m & -\mu/m \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ F_o \cos(\omega t) \end{pmatrix},$$

and if we let $\mathbf{x}(t) = (x(t), v(t))^T$, our initial condition can be written as

$$\mathbf{x}(0) = \begin{pmatrix} x_0 \\ v_0 \end{pmatrix}.$$

Problem 19 The derivative is

$$\mathbf{w}'(t) = \begin{pmatrix} -2e^{-2t} \\ 3\cos(3t) \\ 2 \end{pmatrix}.$$

2.2. **Section 8.2. Problem 13** The proper correspondence is: I-D, II-A, III-B, IV-C.

Problem 14 I've taken the pictures for this problem and the next out of the official solutions manual for our textbook.

Initially, $x(0) = 0$ and $y(0) = 2$, then y decays as x increases, thereafter both x and y oscillate as they decay toward zero.
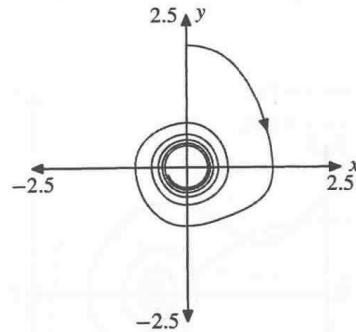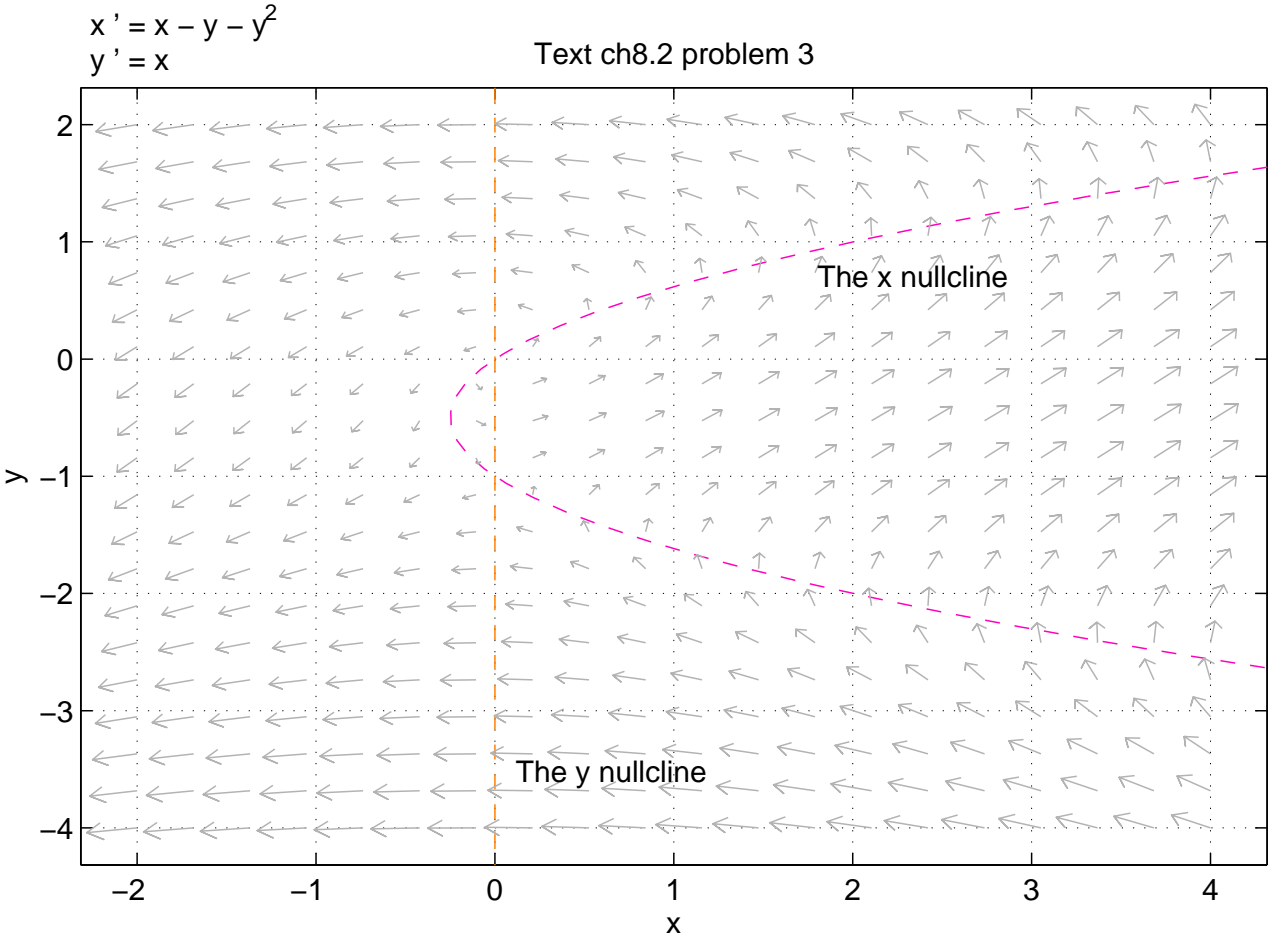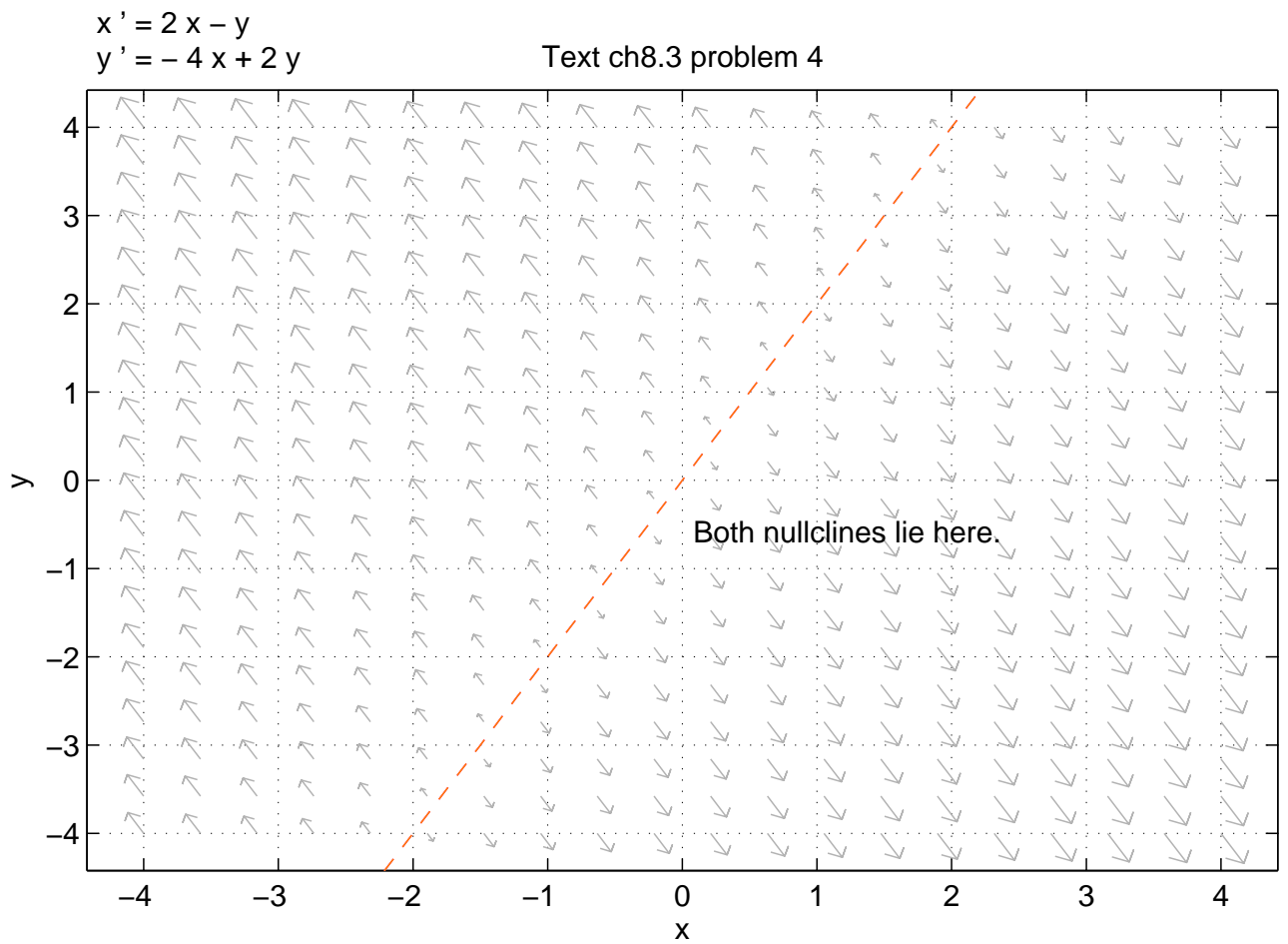
Problem 20

At first x is near 2, with y equaling zero. At first, both x and y decay. Then they both begin to oscillate, all the while decaying toward the origin. It is interesting to note that one component is always zero where the other component is at an extreme. One possible solution follows.
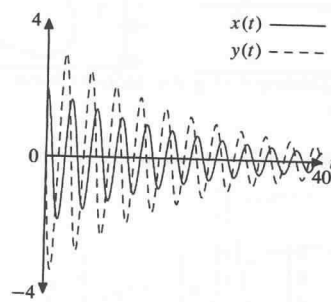
2.3. **Section 8.3. Problem 3** The x-nullcline is given by the curve $y = x - x^3$, and the y-nullcline is given by the curve $x = 0$. I made a plot in `pplane`.


$x' = x - y - y^2$
$y' = x$

Text ch8.2 problem 3

The x nullcline

The y nullcline

Problem 4 The x-nullcline is given by the curve $y = 2x$, and the y-nullcline is given by the curve $y = 2x$. Notice that these coincide. I made a plot in `pplane`.

x ' = 2 x − y
y ' = − 4 x + 2 y

Text ch8.3 problem 4

Both nullclines lie here.

Problem 6 The x-nullcline is given by the curve $x = -y^2$, and the y-nullcline is given by the curve $y = -x$. I made a plot in `pplane`.

x ' = x + y^2
y ' = x + y

Text ch8.3 problem 6

The y nullcline

The x nullcline

y

x
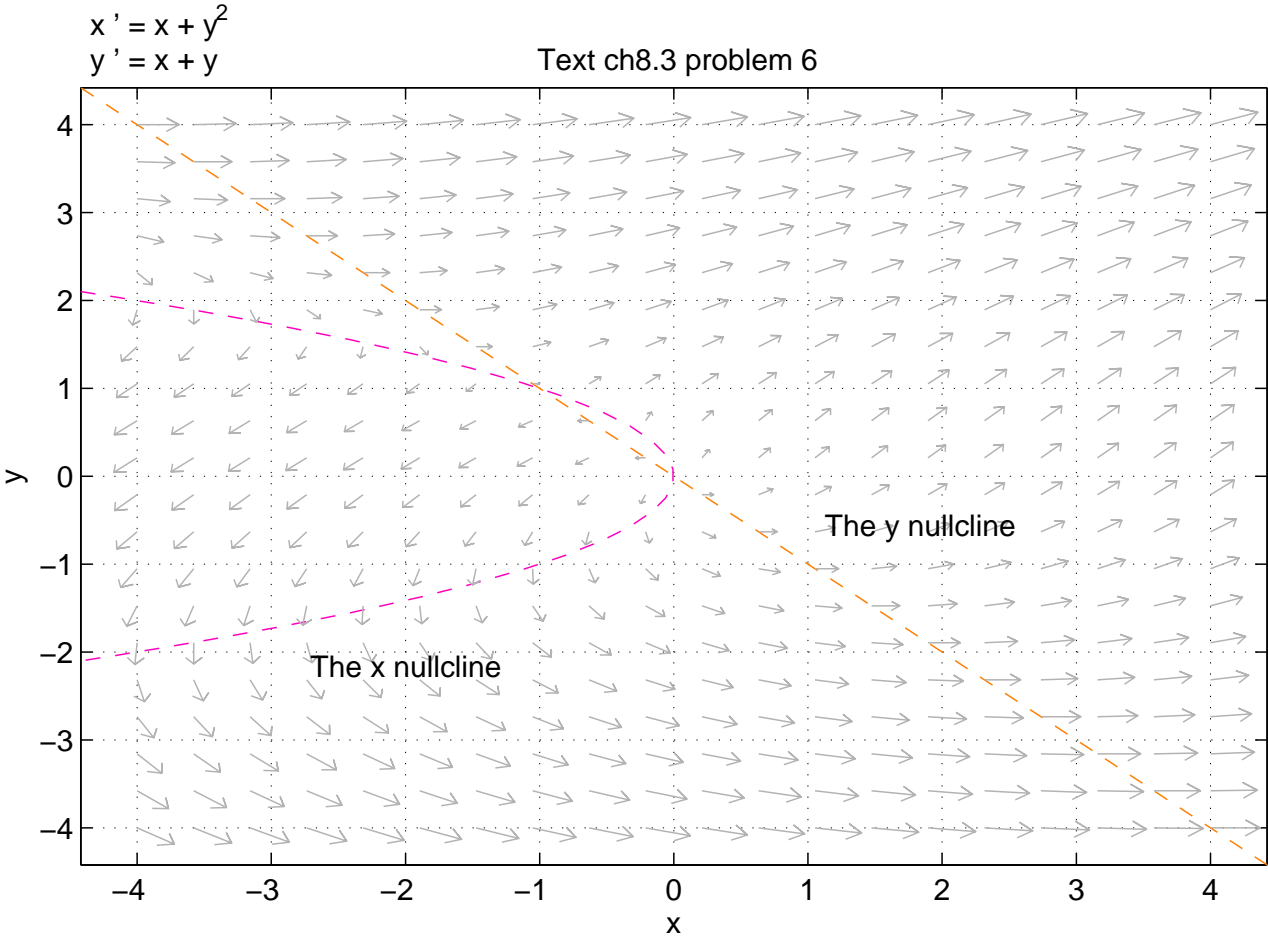
Problem 9 Parts (a) and (b) are just direct checks. Part (c) yields a graph like the one below. Then part (d) follows immediately from the Uniqueness theorem. The system is autonomous, and therefore the solution curves cannot cross in the phase plane. The desired inequality now follows from the picture as the given initial condition is inside the region on the right bounded by the lines $y = \pm x$.
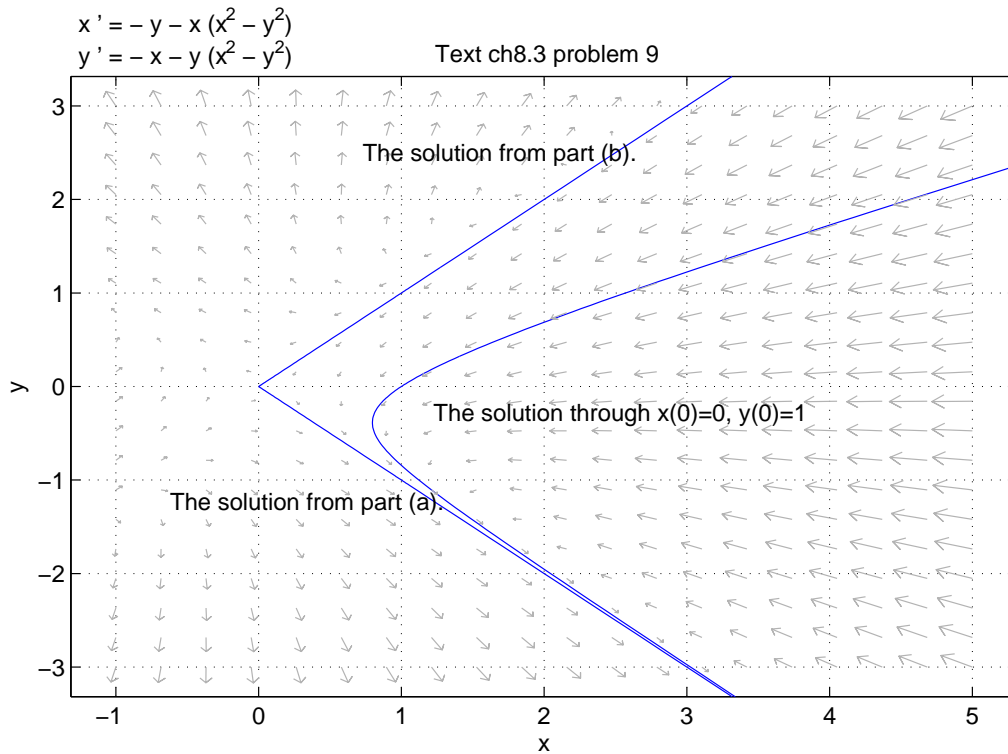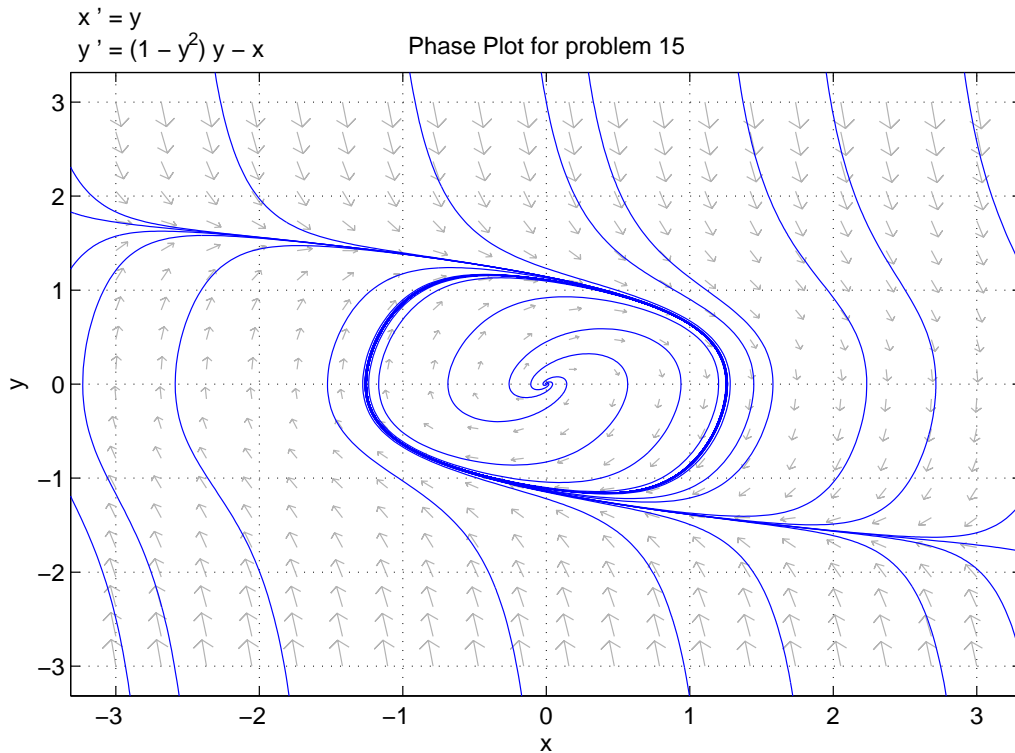

3. EXERCISES FROM THE MANUAL

3.1. **Chapter 7. Problem 15** We are to use `pplane` to make a phase portrait of the system

$$
\begin{aligned}
x' &= y \\
y' &= (1 - y^2)y - x
\end{aligned}
$$

on the box $-3 \le x, y \le 3$. My phase plot looks like the following picture.

x ' = y
y ' = (1 – y²) y – x

Phase Plot for problem 15

3.2. **Chapter 11.** In this section, we are to solve some larger systems of linear equations. MATLAB is a very important tool for this.

Problem 10 Turns out this matrix is invertible. I saw this by computing $\det(A) = 12$. Then use the computer to invert the matrix and write down the solution

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \end{pmatrix}^T.$$

That is, $x_1 = x_3 = x_5 = 0$ and $x_2 = x_4 = 1$.

Problem 16 This matrix is not invertible. I used the computer to find a basis for the nullspace: it is two dimensional.

$$\mathcal{N} = \mathrm{span}\{(-2, -1, 4, 0, 0)^T, (6, -3, 0, 1, 0)^T\}$$

Problem 22 This collection of vectors in linearly dependent. In particular,

$$-5 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + 6 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ -6 \\ -1 \\ -6 \end{pmatrix} = \mathbf{0}$$

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