HOMEWORK # 9 SOLUTIONS

1. Exercises from the Text Chapter 9

1.1. **Section 9.1.** I'll omit the tedious calculations and just post answers here.

Problem 3 The characteristic polynomial is $p_A(t) = (t+2)(t+5)$, so the eigenvalues of A are $-2, -5$.

Problem 8 The characteristic polynomial is $p_A(t) = (6-t)(-9-t)+50 = t^2+3t-4 = (t-1)(t+4)$. Thus the eigenvalues of A are $\lambda = 1, -4$.

Problem 12 The characteristic polynomial is $p_A(t) = (t+1)(t^2 - 5t + 20)$ (expand along the second row of $A - tI$ to compute the determinant). The first factor gives an eigenvalue of -1. The second factor has a negative discriminant, and we get a conjugate pair of complex eigenvalues $\dfrac{5 \pm i\sqrt{55}}{2}$.

Problem 20 We are to solve the system $\mathbf{y}' = A\mathbf{y}$ for $A = \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix}$. The matrix A has trace $T = 0$ and determinant $D = -1$. Thus the characteristic polynomial of A is $p_A(t) = t^2 - 1$. So the eigenvalues of A are ± 1. We row reduce $A - I$ to find that $\mathbf{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to 1. Similarly, we find that $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector for -1. We have found a pair of distinct eigenvalues and their corresponding eigenvectors, so we can simply write down a fundamental set of solutions as follows.

$$\begin{aligned} \mathbf{x}_1(t) &= e^t\mathbf{v}_1 = \begin{pmatrix} e^t \\ e^t \end{pmatrix}, \\ \mathbf{x}_2(t) &= e^{-t}\mathbf{v}_2 = \begin{pmatrix} e^{-t} \\ 2e^{-t} \end{pmatrix}. \end{aligned}$$

Problem 22 In this problem we use the matrix $A = \begin{pmatrix} -3 & 14 \\ 0 & 4 \end{pmatrix}$. The trace is $T = 1$ and the determinant is $D = -12$, so the characteristic polynomial is $p_A(t) = t^2 - t + 12 = (t+3)(t-4)$. We get the following eigenvalue/eigenvector pairs: for the eigenvalue $\lambda_1 = -3$ the corresponding eigenvector is $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, for the eigenvalue $\lambda_2 = 4$ the corresponding eigenvector is $\mathbf{v}_2 = \begin{pmatrix} 14 \\ 1 \end{pmatrix}$. As these are distinct, we are assured that everything works fine and we can write down a fundamental system of solutions directly.

$$\begin{aligned} \mathbf{x}_1(t) =& e^{-3t}\mathbf{v}_1 = \begin{pmatrix} e^{-3t} \\ 0 \end{pmatrix}, \\ \mathbf{x}_2(t) =& \ e^{4t}\mathbf{v}_2 = \begin{pmatrix} 14e^{4t} \\ e^{4t} \end{pmatrix}. \end{aligned}$$

1.2. **Section 9.2.** Again I will omit some of the tedious computations and sketch out the solutions.

Problem 9 We are to solve the initial value problem $\mathbf{y}' = A\mathbf{y}$ for $A = \begin{pmatrix} -5 & 1 \\ -2 & -2 \end{pmatrix}$, $y(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$. The trace is $T = -7$ and the determinant is $D = 12$, so the characteristic polynomial is $p_A(t) = t^2_7 t + 12 = (t+3)(t+4)$. We get the following eigenvalue/eigenvector pairs: for the eigenvalue $\lambda_1 = -3$ the corresponding eigenvector is $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, for the eigenvalue $\lambda_2 = -4$ the corresponding eigenvector is $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. As these are distinct, we are assured that everything works fine and we can write down the general solution directly.

$$\mathbf{x}(t) = C_1 e^{-3t}\mathbf{v}_1 + C_2 e^{-4t}\mathbf{v}_2 = C_1 \begin{pmatrix} e^{-3t} \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 14e^{4t} \\ e^{4t} \end{pmatrix} = \begin{pmatrix} C_1 e^{-3t} + 14C_2 e^{-4t} \\ C_2 e^{-4t} \end{pmatrix}.$$

To satisfy our initial condition we must have

$$C_1 + 14C_2 = 0, \qquad \text{and} \qquad C_2 = -1.$$

So we can solve for $C_2 = -1$ and $C_1 = 14$. This means that our specific solution is

$$\mathbf{x}(t) = \begin{pmatrix} 14e^{-3t} - 14e^{-4t} \\ -e^{-4t} \end{pmatrix}.$$

Problem 23 This time we must find a fundamental set of solutions for the system $\mathbf{y}' = A\mathbf{y}$ where $A = \begin{pmatrix} -4 & -8 \\ 4 & 4 \end{pmatrix}$. This matrix has trace $T = 0$ and determinant $D = 16$, so its characteristic polynomial is $p_A(t) = t^2 + 16$. This has a pair of complex conjugate roots $\lambda_{1,2} = \pm 4i$. We write out the complex solution corresponding to $\lambda_1 = 4i$. The corresponding complex eigenvector is $\mathbf{v}_1 = \begin{pmatrix} -1+i \\ 1 \end{pmatrix}$, so our complex solution is

$$\mathbf{x}_C(t) = e^{4i}\mathbf{v}_1 = \begin{pmatrix} e^{4i}(-1+i) \\ e^{4i} \end{pmatrix}.$$

We use Euler's formula and gather up the real and complex parts to read

$$\mathbf{x}_C(t) = \begin{pmatrix} -\cos(4t) - \sin(4t) \\ \cos(4t) \end{pmatrix} + i \begin{pmatrix} \cos(4t) - \sin(4t) \\ \sin(4t) \end{pmatrix}.$$

We know that our fundamental solutions are just the real and imaginary parts of this complex solution, so we read off

$$\mathbf{x}_1(t) = \begin{pmatrix} -\cos(4t) - \sin(4t) \\ \cos(4t) \end{pmatrix}$$
$$\mathbf{x}_2(t) = \begin{pmatrix} \cos(4t) - \sin(4t) \\ \sin(4t) \end{pmatrix} \quad .$$

Problem 38 This time we must find a fundamental set of solutions for the system $\mathbf{y}' = A\mathbf{y}$ where $A = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix}$. This matrix has trace $T = -4$ and determinant $D = 4$, so its characteristic polynomial is $p_A(t) = t^2 + 4t + 4 = (t+2)^2)$. So we get only one eigenvalue $\lambda = -2$. We find that the nullspace of $A - (-2)I = A + 2I$ is only one-dimensional and is spanned by $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. This will give us one solution very quickly

$$\mathbf{x}_1(t) = e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{-2t} \\ e^{-2t} \end{pmatrix}.$$

We know the other solution is of the form

$$\mathbf{x}_2(t) = e^{-2t}\left(\mathbf{v}_2 + t\mathbf{v}_1\right),$$

where $\mathbf{v}_2$ is a vector chosen so that $(A + 2I)\mathbf{v}_2 = v_1$. If we write $\mathbf{v}_2 = \begin{pmatrix} x \\ y \end{pmatrix}$, then we are looking for a solution to the system

$$(A + 2I)\mathbf{v}_2 = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

That is, we must pick x, y so that $-x + y = 1$. We choose $x = 0$ and $y = 1$. This allows us to write out a second solution in the form

$$\mathbf{x}_2(t) = e^{-2t} (\mathbf{v}_2 + t\mathbf{v}_1) = \begin{pmatrix} te^{-2t} \\ e^{-2t} + te^{-2t} \end{pmatrix}$$

Problem 42 This is the same as the last problem except that we are to use the matrix $A = \begin{pmatrix} 5 & 1 \\ -4 & 1 \end{pmatrix}$. Here the trace is $T = 6$ and the determinant is $D = 9$. Thus the characteristic polynomial is $p_A(t) = t^2 - 6t + 9 = (t-3)^2$, and the only eigenvalue is $\lambda = 3$. The corresponding eigenspace is one dimensional and is spanned by the vector $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. So, as above, we can immediately write down one solution as

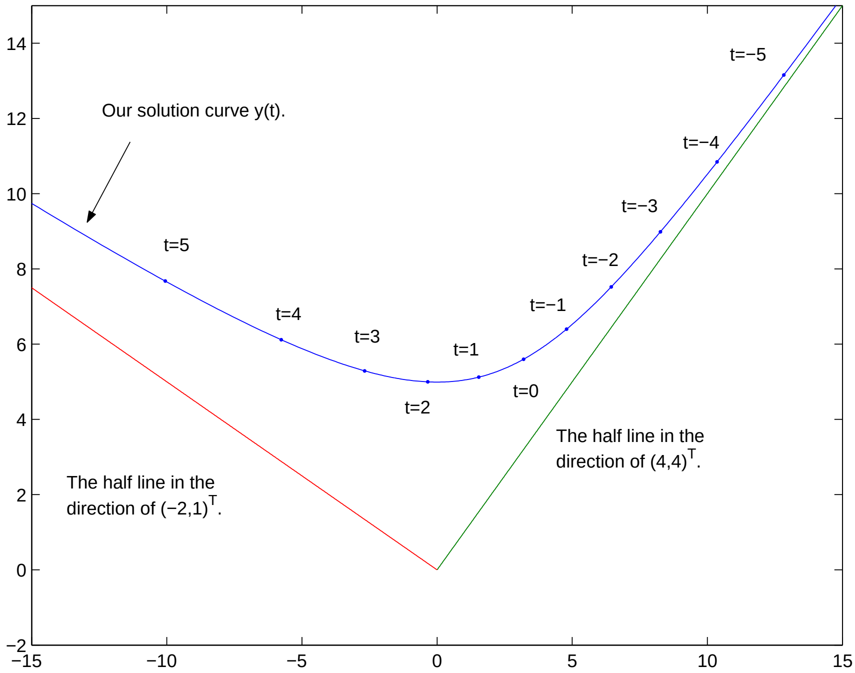
$$\mathbf{x}_1(t) = e^{3t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} e^{3t} \\ -2e^{3t} \end{pmatrix}.$$

For the other solution, we must find a vector $\mathbf{v}_2$ for which $(A - 3I)\mathbf{v}_2 = \mathbf{v}_1$. This matrix equation is equivalent to the single equation $2x + y = 1$, where $\mathbf{v}_2 = \begin{pmatrix} x \\ y \end{pmatrix}$. We pick the solution $x = 0, y = 1$. Therefore, our second solution is

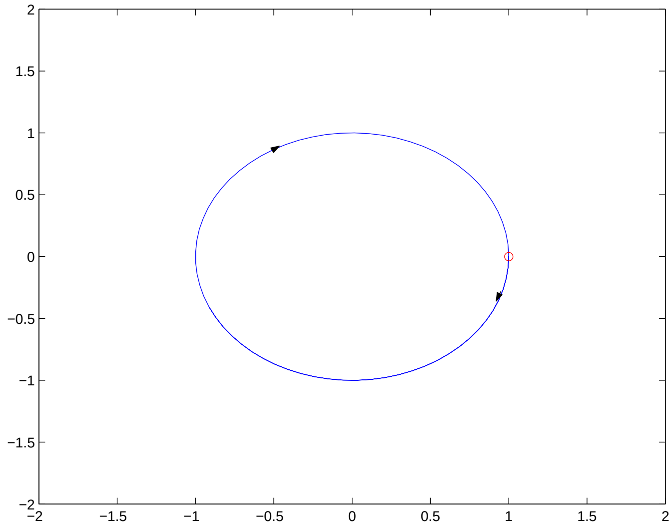
$$\mathbf{x}_2(t) = e^{3t} (\mathbf{v}_2 + t\mathbf{v}_1) = \begin{pmatrix} te^{3t} \\ e^{3t} - 2te^{3t} \end{pmatrix}.$$

1.3. **Section 9.3.**

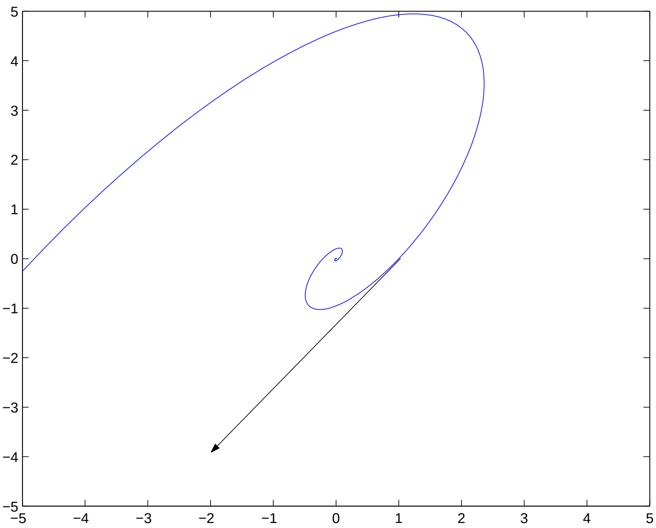
Problem 7 I did all of the computations and plotting in MATLAB. My picture looks like the following.



Problem 17 We are considering the system $\mathbf{y}' = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} \mathbf{y}$. The matrix has trace $T = 0$ and determinant $D = 9$. Thus the characteristic polynomial is $p(t) = t^2 + 9$ which has roots $\lambda = \pm 3i$. The real parts of these are zero, so the equilibrium point is a center. At the point $\mathbf{y} = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$ we get that $\mathbf{y}' = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$. I just went ahead and sketched the solution curve through this point with MATLAB. I can't seem to get the axes to measure up the same. The picture is supposed to be circular. Also, I double checked this picture against the numerical routines in `pplane` and they compared well.



Problem 23 We are consider the system $\mathbf{y}' = \begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix} \mathbf{y}$. This matrix has trace $T = -2$ and determinant $D = 5$. The characteristic polynomial $p(t) = t^2 + 2t + 5 = (t+1)^2 + 4$, and hence the eigenvalues of the matrix are $\lambda_{1,2} = -1 \pm 2i$. As the real parts of these eigenvalues are negative, this equilibrium point is a spiral sink. At the point $\mathbf{y} = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$, the direction vector is $\mathbf{y}' = \begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$. Again, we sketch the solution curve through this point using MATLAB. Also, I double checked this picture against the numerical routines in `pplane` and they compared well.



1.4. **Section 9.4.**

Problem 7 We are to solve the following system.

$$\begin{aligned} x' &= -4x - 5y + 4z \\ y' &= -y + 4z \\ z' &= z. \end{aligned}$$

In matrix form this is

$$\mathbf{w}' = \begin{pmatrix} -4 & -5 & 4 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{w},$$

where $\mathbf{w} = \begin{pmatrix} x & y & z \end{pmatrix}$. Since the matrix in question is upper triangular, it is not difficult to read of the eigenvalues and compute the corresponding eigenvectors. We obtain pairs of $\lambda_1 = -4, \mathbf{v}_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$, $\lambda_2 = 1, \mathbf{v}_2 = \begin{pmatrix} -6 & 10 & 5 \end{pmatrix}^T$ and $\lambda_3 = -1, \mathbf{v}_3 = \begin{pmatrix} -5 & 3 & 0 \end{pmatrix}^T$. Since we have enough to form a fundamental set of solutions, we can write down the general solution easily as

$$x(t) = C_1 e^{-4t}\mathbf{v}_1 + C_2 e^t \mathbf{v}_2 + C_3 e^{-t}\mathbf{v}_3 = \begin{pmatrix} C_1 e^{-4t} - 6C_2 e^t - 5C_3 e^{-t} \\ 10C_2 e^t + 3C_3 e^{-t} \\ 5C_2 e^t \end{pmatrix}.$$

Problem 33 We are to consider the system

$$\begin{aligned} x' &= x \\ y' &= x + y \\ z' &= -10x + 8y + 5z. \end{aligned}$$

The matrix in question is $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -10 & 8 & 5 \end{pmatrix}$. The eigenvalues are 1 and 5. The geometric and algebraic multiplicity of 5 are both one. For 1, the algebraic multiplicity is two, but the geometric multiplicity is one, as the nullspace of $A - tI$ is spanned by the vector $\mathbf{v} = \begin{pmatrix} 0 & 1 & 2 \end{pmatrix}^T$. Thus there are not enough eigenvectors to write out a general solution, and the exercise ends.

Problem 34 This exercise is just like the last one, except that the system in question is

$$\mathbf{y}' = \begin{pmatrix} 2 & 0 & 0 \\ -6 & 2 & 3 \\ 6 & 0 & -1 \end{pmatrix}.$$

The characteristic polynomial of the matrix is $p(t) = \det(A - tI) = (2-t)(2-t)(-1-t)$. So we have a pair of eigenvalues. The eigenvalue -1 has algebraic and geometric multiplicity one. The corresponding eigenspace is spanned by the vector $\mathbf{v}_1 = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix}^T$. The eigenvalue 2 has algebraic multiplicity two. It also has geometric multiplicity two, as the corresponding eigenspace is spanned by the vectors $\mathbf{v}_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T$ and $\mathbf{v}_2 = \begin{pmatrix} 1 & 0 & 2 \end{pmatrix}^T$. This allows us to write down the general solution as

$$x(t) = C_1 e^{-t}\mathbf{v}_1 + C_2 e^{2t}\mathbf{v}_2 + C_3 e^{2t}\mathbf{v}_3 = \begin{pmatrix} C_3 e^{2t} \\ C_1 e^{-t} + C_2 e^{2t} \\ -C_1 e^{-t} + 2C_3 e^{2t} \end{pmatrix}.$$

2. Exercises from the manual chapter 12

Problem 32 Using Matlab's matrix commands, I found the eigenvalue/eigenvector pairs as follows:

$$\lambda_1 = -1, \mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \lambda_2 = 2, \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad \lambda_3 = -3, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Because we can write

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 2\mathbf{v}_1 - \mathbf{v}_2 + 0\mathbf{v}_3,$$

we know that the answer can be written as

$$\mathbf{y}(t) = 2e^{-t}\mathbf{v}_1 - e^{2t}\mathbf{v}_2 = \begin{pmatrix} e^{2t} \\ 2e^{-t} - e^{2t} \\ 2e^{-t} - e^{2t} \end{pmatrix}.$$

Problem 33 Again, I used MATLAB's `eig` command to find the eigenvalues and eigenvectors of the matrix. Some of them are messy, and to get workable numbers (i.e. integers) I used the `poly`, and `null` commands, too. It is especially useful to use `null(·,'r')` command.

I get the following pairs:

$$\lambda_1 = 4, \mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \quad \lambda_2 = -3, \mathbf{v}_2 = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \quad \lambda_3 = -3, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.$$

This system is a bit easier to solve because the algebraic multiplicity and the geometric multiplicity of the eigenvalue -3 match at two. Since

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 5\mathbf{v}_1 - \frac{3}{2}\mathbf{v}_2 - \frac{9}{2}\mathbf{v}_3,$$

we know that the answer can be written as

$$\mathbf{y}(t) = 5e^{4t}\mathbf{v}_1 + e^{-3t}\left(-\frac{3}{2}\mathbf{v}_2 - \frac{9}{2}\mathbf{v}_3\right) = \begin{pmatrix} 10e^{4t} - 9e^{-3t} \\ 5e^{4t} - 3e^{-3t} \\ 5e^{4t} - 9e^{-3t} \end{pmatrix}.$$

Problem 34 This exercise is similar to the last two. The eigenvalues are 1 and $2 \pm i$. The eigenspace for 1 is two dimensional and spanned by

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ -1 \\ 2 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 4 \end{pmatrix}.$$

To deal with the complex eigenvalues, we just proceed as normal (as if complex numbers didn't bother us) until the very end. The eigenvectors corresponding to $2+i$ and $2-i$ are, respectively,

$$\mathbf{v}_3 = \begin{pmatrix} 2-i \\ 1-3i \\ 0 \\ 5 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_4 = \begin{pmatrix} 2+i \\ 1+3i \\ 0 \\ 5 \end{pmatrix}.$$

I then used MATLAB to find that

$$\begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2}\mathbf{v}_1 + \frac{11}{2}\mathbf{v}_2 + \frac{-21+2i}{10}\mathbf{v}_3 + \frac{-21-2i}{10}\mathbf{v}_4.$$

Hence, we can write out our solution as

$$\begin{aligned}\mathbf{x}(t) =& e^t\left(\frac{1}{2}\mathbf{v}_1 + \frac{11}{2}\mathbf{v}_2\right) + e^{(2+i)t}\frac{-21+2i}{10}\mathbf{v}_3 + e^{(2-i)t}\frac{-21-2i}{10}\mathbf{v}_4 \\ =& e^t\begin{pmatrix}11\\5\\1\\22\end{pmatrix} + e^{(2+i)t}\frac{-21+2i}{10}\mathbf{v}_3 + e^{(2-i)t}\frac{-21-2i}{10}\mathbf{v}_4\end{aligned}$$

To get somewhere with the last two terms, one can just plod on through, or notice that the third term is the conjugate of the second. Hence the sum of these is equal to twice the real part of the second term. This helps us to compute that

$$\mathbf{x}(t) = e^t\begin{pmatrix}11\\5\\1\\22\end{pmatrix} + e^{2t}\begin{pmatrix}-8\cos(t) - 5\sin(t)\\ -3\cos(t) - 13\sin(t)\\ 0\\ -21\cos(t) - 2\sin(t)\end{pmatrix}$$

This was a pretty complicated computation, so I checked it by asking MATLAB to do the following.

```
syms t real;

A= (type in matrix)

f= (type in the expression above as a vector of functions)

z=diff(f);

w=A*(f);

z-w
```

This returns a zero column vector, so we are happy.