MATH 211 LECTURE 10

Mathematical Modelling, part II

1. Two more standard types of models

The remainder of the third chapter in the text is about other possible applications. One section is about simple interest compounding, the other is about equations for electrical circuits. I just want to mention these briefly.

1.1. compound interest. Basic idea is that

$$P(t+\Delta t) - P(t) = (\textit{amount in per time unit })\Delta t - (\textit{ amount out per time unit })\Delta t.$$

And hence, $P'(t) = (\text{interest rate})P(t)\ldots$

Then we can adjust for lots of different scenarios where we accrue interest, make regular payments, take out money regularly, etc. . . It is worth noting that these models usually use the hypothesis of continuously compounded interest. No bank really does this, but one can still get reasonable models–especially good for daily compounding.

Most of these equations lead to exponential models, or mixes of them. The examples of the text are like this.

Of course, the basic message of the examples that are put together in this section is that retirement is a lot closer than you think, and that you need to start planning creatively.

One important message is that technically one must always start from the basic principles when designing a model. But in principle, it is not always required that we go back to the pre-differential equation step and take the limit. Sometimes it is pretty clear what will happen, and this saves work.

1.2. electrical circuits. There is a decent description in the text of the output from the theory of electrical circuits in physics. The interesting part is that a typical resistor-capacitor-inductor circuit yields a second order differential equation in the current I like

$$A\frac{d^2I}{dt^2} + B\frac{dI}{dt} + CI + D = 0$$

This isn't something we have learned to handle yet, but in certain cases, we can do it. If A=0, we are in a first order linear equation–which we can do. (this corresponds to no inductor)

A similar trick is to handle the case where C=0. Here we make a substitution: $S = dI/dt$. Then the equation is a first order equation in S. So we solve for S, and then integrate it again to find I. This is a useful trick.

In a way, this is a special case of turning the equation into a system—possibly discuss this. Perhaps also discuss making equation an autonomous system because it is really the same trick.

2. "NON-DIMENSIONALIZING" AN EQUATION

This is a weird topic the first few times you see it. The basic idea is that sometimes the units get in the way of understanding.

It can be difficult to decide what constitutes "small" or "large" for some measurements.

Often, changing the time scale of a problem will change all the constants at once, and you get an equation that is just as good as the original.

The thing to do here is to rescale everything so that the math is easier. This is a bit of an art form, but when done properly will result in fewer constants floating around, and it will often be clear what quantities one needs "small" or "large" to get typical behaviors. Read part out of Strogatz.

Just need to see some examples. We consider the logistic equation.

$$\frac{dP}{dt} = rP(t)(1 - \frac{P(t)}{K}) = rP(t) - \frac{rP(t)^2}{K}.$$

We let a and b be constants (to be chosen later) and we consider new variables $a\tau = t$ and $bN = P$. The trick is that a is supposed to have units of "time" and b is supposed to have units of "population", i.e. "fish", but N and τ are unitless quantities. Think of them as concentrations. We then use the chain rule and a whole bunch of substitutions to rewrite our equation as

$$\frac{b}{a}\frac{dN}{d\tau} = brN - \frac{rb^2}{K}N^2$$

We then multiply through by a/b to get

$$\frac{dN}{d\tau} = arN - \frac{abr}{K}N^2$$

This is where we see how to make our choices. The equation would look a lot nicer if the constants here were *ones*. So we choose $a = 1/r$. Notice that the units have worked. This constant is often called the *characteristic time scale*. It is somehow the naturally defined length of time that the differential equation likes... And we choose $b = K$. Again, the units have worked! Now the "maximum" population is 1. We get some uniform notion ahead of time of what is a small number (close to zero) and what is a big number (close to one) for the population. Again, this works with the concentration analogy.

Also, the end result is that our equation looks like

$$dN/d\tau = N - N^2$$

which is prettier (i.e. with fewer arbitrary constants).