# MATH 211 LECTURE 11

## Numerical Methods, Part I

Today we begin study of the numerical methods for ordinary differential equations. Our goal is to see the most basic tools available and to discuss the general types of difficulties one encounters. A good craftsman knows his tools.

In general, we shall discuss what are called *numerical solvers.* These don't actually produce solutions, just sequences of numbers which approximate the solutions.

### 1. The general procedure

We will start with an initial value problem like $y' = f(y,t)$, $y(t_0) = y_0$. The idea is to specify a sequence of times $t_0, t_1, \ldots, t_k$ and compute a sequence of approximate values to the real solution $y_0, y_1, \ldots, y_k$. We want to keep each of the differences $y(t_i) - y_i$ as small as we can.

The starting point for our work is the observation that if $y(t)$ is a solution to the initial value problem, then we have

$$y(t) = y_0 + \int_{t_0}^{t} f(y(s), s)\ ds = y_0 + \int_{t_0}^{t} f(y(s), s)\ ds$$

The key is to use our knowledge of calculus to find good ways of approximating the integral on the right.

### 2. Euler's method

In this method, we approximate the integrand $f(y(s), s)$ by its value at the left hand endpoint. Draw picture on a slope field. The resulting procedure is as follows.

**1:** Fix a *step size*, $h$.

**2:** Given a point $(t_i, y_i)$, we obtain the next point by the rules

$$\begin{aligned} t_{i+1} &= t_i + h, \\ y_{i+1} &= y_i + h \cdot f(y_i, t_i). \end{aligned}$$

So, given a problem, we must choose a step size, divide up the interval we care about into the requisite number of pieces, and then use the above rules to pass from one point to the next.

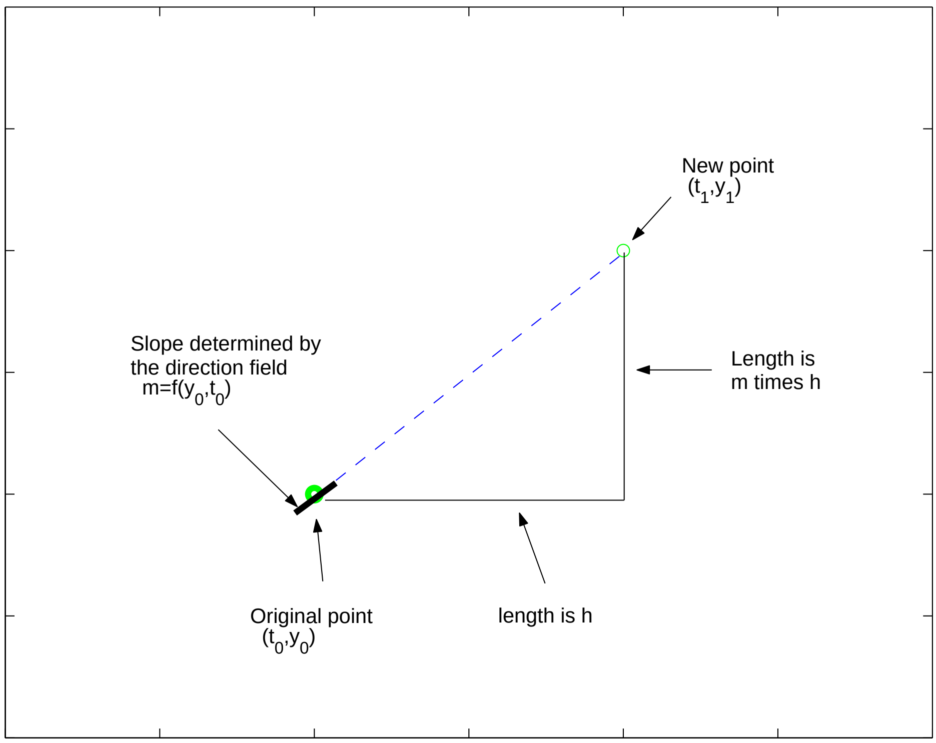
**Example.** Use Euler's method to approximate the solution to $y' = y + t$, $y(0) = 1$ on the interval $[0, 1]$. Use a step size of $h = 1/4$. Compare with a step size of $h = 1/8$, and with the actual solution.

I've tabulated the results from the example above with $h = 1/4$ just below. The same can be done for $h = 1/8$, but the table is larger.

Euler's method for $y' = y + t$, $y(0) = 1$, $h = .25$.

| step $k$ | $t_k$ | $y_k$ | $f(y_k, t_k)$ | $h \cdot f(y_k, t_k)$ | Actual $y(t_k)$ | percent error |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | .25 | 1 | 0 |
| 1 | .25 | 1.25 | 1.5 | .25 | 1.3180508 | 5.16 |
| 2 | .5 | 1.625 | 2.125 | 0.53125 | 1.7974425 | 9.59 |
| 3 | .75 | 2.15625 | 2.90625 | 0.7265625 | 2.4840000 | 13.19 |
| 4 | 1 | 2.8828125 | | | 3.43654 | 16.11 |

Graphically, Euler's method is this procedure:



## 3. Runge-Kutta methods

In these methods, we use other ways to approximate the integral. There are many different flavors, but they all exhibit the "predict and correct" philosophy.

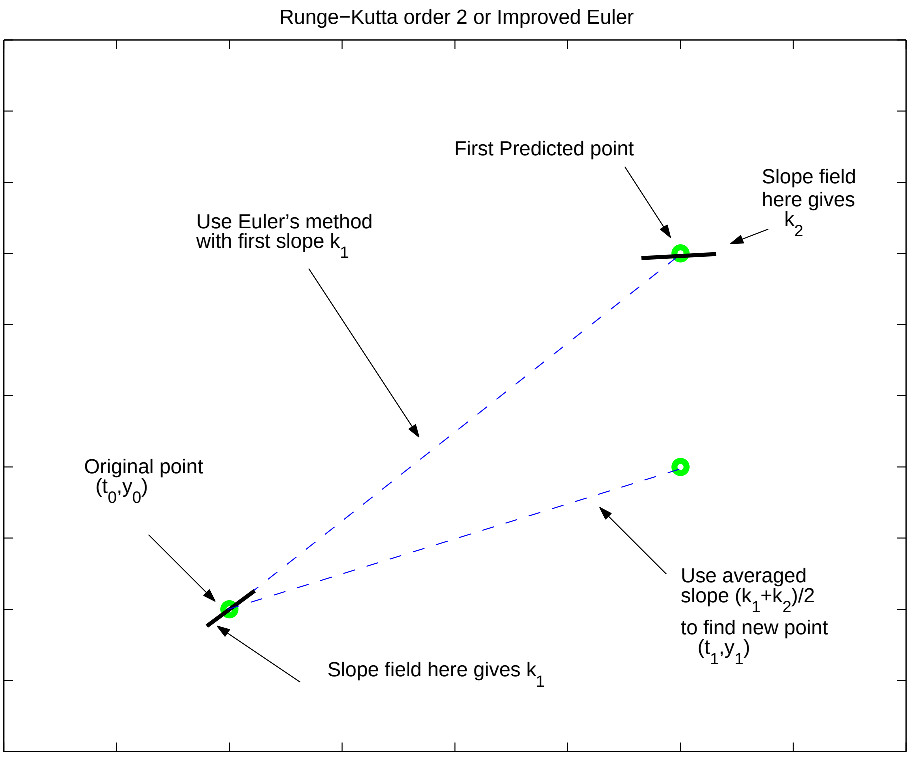
### 3.1. **Order Two.**

This method is also called the improved Euler method. Here we use the *trapezoid rule* for integration. The procedure is as follows. (Draw picture.)

**1:** Fix a *step size*, $h$.
**2:** Given a point $(t_i, y_i)$, we obtain the next point by the rules

$$\begin{aligned} t_{i+1} &= t_i + h, \\ k_1 &= f(y_i, t_i) \\ k_2 &= f(y_i + h \cdot k_1, t_i + h) \\ y_{i+1} &= y_i + h \cdot \frac{k_1 + k_2}{2}. \end{aligned}$$

Graphically, the second order Runge-Kutta method is this procedure:



3.2. **Order Four.** There is more than one method of "order four" as that is a technical term to be defined below. But the first one to look at comes from using Simpson's rule for approximating the integral. The procedure is as follows. (Draw picture.)

**1:** Fix a *step size*, $h$.

**2:** Given a point $(t_i, y_i)$, we obtain the next point by the rules

$$\begin{aligned} t_{i+1} &= t_i + h, \\ k_1 &= f(y_i, t_i) \\ k_2 &= f(y_i + k_1 \cdot h/2, t_i + h/2) \\ k_3 &= f(y_i + k_2 \cdot h/2, t_i + h/2) \\ k_4 &= f(y_i + k_3 \cdot h, t_i + h) \\ y_{i+1} &= y_i + h \cdot \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}. \end{aligned}$$

One can draw a picture for this method just like the above two. It is a good exercise to be able to draw it from the algorithm. Also, one should be able to use the picture to recreate the algorithm.

**Example.** We can re-examine the problem above with the RK2 and RK4 methods. We use step sizes of $h = 1/4$ and $h = 1/8$ again. One should use a computer as the work is tedious.