MATH 211 LECTURE 13

Matrix algebra, vectors and linear equations

We begin our study of linear algebra with the basics. We shall try to take the shortest route to the results we need to understand systems of differential equations.

1. Matrices and vectors

Matrices and vectors are just arrays of numbers. They are tailored to help us solve problems involving linear equations, which we shall see later today. But before they can become useful, we have to know what they are and how to play with them.

An $m \times n$ *matrix* is an array of numbers which has m rows and n columns. Abstractly, the general $m \times n$ matrix can be written like

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & \cdots & a_{mn} \end{pmatrix}$$

The first index on the symbol a_{ij} tells you that the number lives in the i-th row, and the second symbol tells you that it lives in the j-th column. This specifies where in the matrix to put the number. Sometimes we shall denote a matrix by the symbol $A = (a_{ij})$, where the index i runs from 1 to m and the index j runs from 1 to n.

Examples: give 2×3, 3×2, 4×4, 2×4 matrices as examples. just use arbitrary numbers. Be sure to drop in some irrationals and even transcendentals. What is a 1×1 matrix?

A *row vector* is a matrix which is just a single row with many columns. That is, a row vector is a $1 \times n$ matrix like

$$v = \begin{pmatrix} 7 & 5 & 34 & \pi & e^2 & 0 \end{pmatrix}$$

A *column vector* is a matrix which is just a single column with many rows. That is, a column vector is a $n \times 1$ matrix like

$$w = \begin{pmatrix} 7 \\ 5 \\ 34 \\ \pi \\ e^2 \\ 0 \end{pmatrix}$$

We shall have slightly more use for column vectors than row vectors. It is probably safe to assume that if I just say vector, I mean a column vector.

A matrix which has size $n \times n$ is called a *square matrix of size n*. The matrix $E = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is square of size 2.

Notice that my matrices have round brackets. This is not for any particular reason other than that I think they look better this way. Sometimes I will forget and write them with curly braces or with square brackets.

2. Matrix Algebra

Since matrices are just bundles of numbers, it stands to reason that one can do some algebraic manipulation with them like numbers. Some care is required to get all of the definitions correct, and trying to divide can be a disaster until you know more.

2.1. addition.

Given two matrices A and B of the *same size* we can add rather easily. Just add component by component.

For example, if

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ and } \quad B = \begin{pmatrix} 3 & -1 \\ \pi & 7 \end{pmatrix},$$

then it is easy to compute that

$$A + B = \begin{pmatrix} 3 & 0 \\ 1+\pi & 7 \end{pmatrix}.$$

Warning! If your matrices have different sizes, then matrix addition doesn't make any sense. It is 'undefined'.

Examples: Make up some problems on the spot with various numbers for the students to try together. Be sure some examples are not square.

2.2. vector addition.

You have probably seen some addition of vectors before. It is worth pointing out that this definition above is the same as the one you know. Often, we shall use a geometric interpretation of vector addition.—give picture of this.

2.3. scalar multiplication.

When considering vectors and matrices, regular numbers are often called *scalars*, just to give them a name to distinguish them from the other types of objects running around. Any matrix can be multiplied by any scalar. The idea is to multiply the scalar quantity to all of the entries of the matrix.

Example: Consider the 2×3 matrix

$$C = \begin{pmatrix} 3 & 0 & -5 \\ 0 & -2 & 1 \end{pmatrix}.$$

Then

$$7 \cdot C = \begin{pmatrix} 21 & 0 & -35 \\ 0 & -14 & 7 \end{pmatrix}.$$

Give students an example or two to work on.

2.4. **Matrix subtraction.** Note that $A - B$ is just $A + (-1) \cdot B$. Also, the *zero matrix* of size $m \times n$ acts just like zero for numbers. It is the matrix O with all entries equal to zero. Then $A + O = A$, and $A - A = O$, etc.

2.5. **Matrix Multiplication.** This is the most important operation for us as it is the part most important to our future work with differential equations. We can multiply a pair of matrices only under constraints on their size. Consider an $m \times n$ matrix $A = (a_{ij})$ and a $n \times p$ matrix $B = (b_{jk})$. We define the matrix product of A and B to be the $m \times p$ matrix

$$C = A \cdot B = (c_{ik})$$

where $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$. This looks a bit weird the first time, but it isn't so bad.

Describe the row times column "zip" for multiplying matrices. Notice how it requires that "the sizes match up in the middle". Do some examples. Matrix multiplication is 'not defined' for pairs which do not match up in size properly.

Of course, if the matrices are both square of the same size n, then we can take their product in both orders! That is, we can consider $A \cdot B$ and $B \cdot A$. The interesting part is that these two operations need not be the same.

Example Use $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ to illustrate this.

Two matrices A and B which are square of size n are said to *commute* if we have $A \cdot B = B \cdot A$. The matrices in the last example do not commute, but the matrix A does commute with $C = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$. Do this in class.

2.6. **The identity matrix.** The *identity matrix* of size n is the matrix

$$I = I_n = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

which has zeros down the *diagonal* (i.e those entries a_{ij} with $i = j$.

We say that a square matrix A of size n is *invertible* if there is another square matrix B also of size n such that $A \cdot B = I_n$. In this case, we also get that $B \cdot A = I_n$ and A and B commute. If A is invertible, then we denote its inverse by the symbol A^{-1}.

Not all matrices are invertible. For example, the matrix

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

doesn't have a multiplicative inverse. That is, there is no matrix C such that $C \cdot D = I$

Compare the operations we know to those of the set of integers.

2.7. **A new operation.** Given an $m \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & \dots & a_{mn} \end{pmatrix},$$

we can define the *transpose* of A to be the $n \times m$ matrix

$$A^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} & \dots & \dots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \dots & \dots & a_{m2} \\ a_{13} & a_{23} & a_{33} & \dots & \dots & a_{m3} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \dots & \dots & a_{mn} \end{pmatrix}$$

For example, if $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$, then $A^T = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$. What is the transpose of the row vector v from above?

3. LINEAR EQUATIONS

The wonderful part of what we have set up is that now we have a very compact way of writing systems of linear equations. Discuss how to write an equation in two or three unknowns as a matrix equation like $A \cdot x = b$ where A is a matrix of the appropriate size, and x and b are vectors. A solution becomes a vector of the same size as x.

Do some explicit examples to see how this works.

Example: The system of linear equations in three unknowns x, y, z given by

$$\begin{aligned} 3x + 2y \ -z &= 8 \\ 12x + 7y + 5z &= 1 \\ x - y \ -z &= 0 \end{aligned}$$

can be reinterpreted in matrix notation as

$$\begin{pmatrix} 3 & 2 & -1 \\ 12 & 7 & 5 \\ 1 & -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \\ 0 \end{pmatrix}.$$

Or $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$, where

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} 3 & 2 & -1 \\ 12 & 7 & 5 \\ 1 & -1 & -1 \end{pmatrix} \\ \mathbf{x} &= \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ \mathbf{b} &= \begin{pmatrix} 8 \\ 1 \\ 0 \end{pmatrix} \end{aligned}$$

3.1. **Matrix Division.** We haven't discussed this yet. Well, we wish we could divide. It would make our lives easier. Truth be told, it is not possible to "divide" all of the time. We really think about division as *multiplication by the inverse* and then it seems possible. But there are troubles. For example, the matrix $D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ above doesn't have a multiplicative inverse.

If we could show that A was invertible, then we could solve the system of equations by writing $\mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$. Wouldn't that be nice? We'll spend a whole lot of time trying to work around this problem. That is, when does a square matrix have an inverse? If it does, how do we find it? If the matrix we get isn't square, what do we do?