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Subspaces, etc.

Recall our proposition from before spring break that a system of linear equations $A \cdot \mathbf{x} = \mathbf{b}$ has a solution set which can be described in terms of the solutions to the associated homogeneous equation $A \cdot \mathbf{x} = \mathbf{0}$.

So we take up the study of homogeneous systems with the hope of gaining some insight on the general problem. (In fact, when we get around to applying this knowledge to differential equations, this case will be important.)

The *nullspace* of a matrix A is the solution set to the corresponding homogeneous system of equations $A \cdot \mathbf{x} = \mathbf{0}$.

Proposition: The nullspace $\mathcal{N}$ of an $m \times n$ matrix is a *vector subspace* of $\mathbb{R}^n$. That is, it satisfies the following properties:

(1) $\mathcal{N}$ is not empty. (it contains $\mathbf{0}$.)
(2) For any two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathcal{N}$, we have $\mathbf{x} + \mathbf{y}$ is in $\mathcal{N}$.
(3) For any real number c and any $\mathbf{x}$ in $\mathcal{N}$, the vector $c \cdot \mathbf{x}$ is also in $\mathcal{N}$.

The key point is that the regular operations apply here and the results don't leave the nullspace. These properties basically say that $\mathcal{N}$ has the same type of structure as the vector space $\mathbb{R}^n$, it is just smaller. By using the second and third properties, it is easy to see that the following property also holds.

Proposition: If $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are in $\mathcal{N}$ and $a_1, \ldots, a_k$ are real numbers, then the vector $a_1 \cdot \mathbf{x}_1 + \cdots + a_k \cdot \mathbf{x}_k$ is also in $\mathcal{N}$.

The type of sum in the last proposition is called a *linear combination* of the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$.

Examples: Find the nullspaces of

$$\begin{pmatrix} -1 & -2 & 3 \\ 1 & 2 & 1 \\ 2 & 4 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 0 & 4 & 2 & 0 \end{pmatrix}$$

In all of these examples, the nullspace is given by linear combinations of just a few vectors. To formalize this we use the definition.

Suppose that we are given vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$. The *span* of these vectors is the set of all of their linear combinations. Sometimes this is denoted $\operatorname{span}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$. It is not difficult to see that the span of a set of vectors is also a subspace of $\mathbb{R}^n$.

It would be nice to be able to pick a smallest subset of the nullspace which spans it. This will enable us to avoid repetition.

A set of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$ is called *linearly independent* if the only way to make a linear combination of them which represents the zero matrix is with all coefficients equal to zero. That is, if the only solution to the equation

$$a_1 \cdot \mathbf{x}_1 + \cdots + a_k \cdot \mathbf{x}_k = \mathbf{0}$$

is the trivial solution $a_1 = a_2 = \cdots = a_k = 0$. A set which is not linearly independent is called *linearly dependent*. Note that a set is linearly dependent exactly when one of the vectors lies in the span of the others. This means that there is redundancy in describing the span.

Examples The set $\{v_1 = (7,1,9)^T, v_2 = (2,-1,9)^T, v_3 = (-3,2,-16)^T\}$ in $\mathbb{R}^3$ is linearly dependent because $-v_1 + 17v_2 + 9v_3 = 0$. The set $\{w_1 = (2,1,3)^T, w_2 = (1,-3,1)^T, w_3 = (1,7,1)^T\}$ is linearly independent. To see this, note that $\operatorname{span}(w_2, w_3) = \operatorname{span}((1,0,1)^T, (0,1,0)^T)$. Hence $\operatorname{span}(w_1, w_2, w_3) = \operatorname{span}((0,0,1)^T, (1,0,1)^T, (0,1,0)^T) = \operatorname{span}((0,0,1)^T, (0,0,1)^T, (0,1,0)^T)$ is all of $\mathbb{R}^3$.

This idea gives us a way to more efficiently describe our nullspaces. Fix some subspace V in $\mathbb{R}^n$. A set of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$ in V is a *basis* of V if it is a linearly independent set and $V = \operatorname{span}(a_1 \cdot \mathbf{x}_1 + \cdots + a_k \cdot \mathbf{x}_k)$.

Important note: A basis is not unique. There are usually lots and lots of different bases for a given subspace. This means your answer can look very different, but actually be the same. Think of a basis as a basic set of allowed directions. Use can use north and east, or northwest and south, etc. Using a different basis was important in our understanding the second example above.

The dimension of a subspace is defined to be the number of elements in a basis. It is a chore to prove that this number is the same for all bases. In all cases we care about, this number will correspond to the "natural" notion of dimension.

This leaves us with another problem, though. When can we say that a set of vectors is linearly independent? There is a test.

Proposition: Suppose that $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are vectors in $\mathbb{R}^n$. Let $X = [\mathbf{x}_1, \ldots, \mathbf{x}_k]$ be the matrix with columns equal to the $\mathbf{x}_i$'s.

(1) If the nullspace $\mathcal{N}(X)$ is the trivial subspace $\{\mathbf{0}\}$, then the vectors are linearly independent.
(2) If the nullspace contains a nontrivial vector, then the set $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is linearly dependent.

The proof of this is to restate the definitions of dependence and independence along with the definition of matrix multiplication. Key observation: The vector equation $a_1 \cdot \mathbf{x}_1 + \cdots + a_k \cdot \mathbf{x}_k = \mathbf{0}$ is equivalent to the matrix equation $X \cdot (a_1, \ldots, a_k)^T = \mathbf{0}$.

This changes the problem into one we know how to solve: Use row reduction. But that is not very satisfying, as row reduction can take a long time. Next time we'll find a tool for doing this better.

The long and the short of this discussion is the following: These tools help us describe the solution sets to homogeneous linear systems effectively. As this is part of describing the solution set to an arbitrary problem, we are happy to have the information.