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Determinants

Our next goal is to find a good criterion for when a square matrix is invertible. What we do is to construct a function $\{$ square matrices$\} \to \mathbb{R}$ which tells us when a matrix is invertible, it is called the *determinant*, and we often use the notation $\det(A)$ for the determinant of a square matrix A.

This process will relate to our program of solving equations, but the determinant is really a property of A, not of the whole matrix equation $A \cdot \mathbf{x} = \mathbf{b}$. Perhaps it is easiest to relate all of the properties to solving a homogeneous system, which doesn't have $\mathbf{b}$ in it anyway. The important thing is that our matrix be able to tell us when there is some redundancy in the equations.

The idea is that a matrix is non-singular exactly in the case that when put in row echelon form it has no zeros on the diagonal. This will mean that every diagonal entry is a pivot, and hence the reduced row echelon form is $\mathbf{I}$. So the key is really being able to get your pivots on the diagonal. If you do some long computations (this is really best done with a mathematical induction), you get out the determinant function that we want. That is, take a general matrix $A = (a_{ij})$, row reduce it so that it is *upper triangular*. That is, perform row operations that zero out all the entries below the diagonal. Then having none of the diagonal entries equal to zero is equivalent to having their product non-zero. So we take this product and call it the determinant. This is not too hard to see in two or three dimensions–do it. This process leads us to the following definition.

Definition: Let $A = (a_{ij})$ be a square matrix of size n with real number entries. The determinant of A is the real number

$$\det A = \sum_{\sigma} (-1)^{\sigma} \prod_{i=1}^{n} a_{i\sigma(i)}.$$

Where the sum runs over all *permutations* σ of the set $\{1, 2, \ldots, n\}$. discuss this briefly. Note that this fancy definition is practically useless to us.

From our construction, we get immediately the following important result.

Theorem: A square matrix A is invertible if and only if $\det(A) \neq 0$.

Note that the nullspace of an invertible matrix must be the trivial subspace $\{\mathbf{0}\}$.

How to compute a determinant

Our first step is to notice that the determinant of an upper triangular matrix is the product of the diagonal entries. This and the above discussion helps motivate our approach to computation.

Basic Definition: Not so pleasant, especially for large matrices.

Row reduction: Apply row operations to put the matrix into upper triangular form. The determinant of the result is easy to compute. The other side is that we have to keep track of the operations we use, as they can change the determinant! Switching two rows multiplies the determinant by -1, multiplying a row by a scalar multiplies the determinant by the same scalar, and adding a multiple of a row to another row leaves the determinant alone. We can always do regular row reduction using just the first and third operations, so things aren't bad.

expansion along a row: Pick a row and do an alternating sum of determinants of *minors*. A minor is the matrix built out of picking one row and one column and removing them. Whether we start the alternating sum with a plus or minus needs some care.

Examples: Compute a couple using the last two methods. Use the examples of

$$\begin{pmatrix} 5 & 6 & 4 \\ -4 & -9 & -8 \\ 4 & 6 & 5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & -1 & 3 & 4 \\ 0 & 2 & -2 & 0 \\ -1 & 2 & 0 & 0 \\ -1 & 3 & 1 & 2 \end{pmatrix}$$

Important properties of the determinant

products: For two square matrices A, B of the same size, $\det(AB) = \det A \cdot \det B$.

inverses: If A is a nonsingular square matrix, then $\det A^{-1} = 1/\det A$.

transposes: $\det A^T = \det A$.

This last property allows us to use column operations just as well as row operations for the second method of computing, or to expand along a column instead of a row. This gives us lots of flexibility in computing determinants: Pick a row or column with lots of zeros!

One more thing: One can show that for a square matrix A which has columns given by the vectors $v_1, \ldots, v_k$, then $\det(A)$ is equal to the n-dimensional volume of the parallelipiped with sides given by the v_is. This is an important fact often used in multivariable calculus.

Overview of the program so far

We now know how to solve a system of linear equations pretty effectively. First we put the equations into augmented matrix form. We can find solutions by applying the Gaussian elimination process to produce an equivalent, but easier, set of equations. In the end, we expect to get a nice description for the solution set like

$$\text{particular solution} + \mathcal{N}$$

where $\mathcal{N}$ is the null space of the matrix (the solution set to the associated homogeneous problem). To finish the description, we need to find a convenient basis for the nullspace. We usually get one out of the Gaussian elimination process, but maybe we want another one for some reason.

Alternatively, one can start with the homogeneous problem. That is, find the nullspace and a good basis for it. If the matrix is non-singular, then we can compute its inverse and use this to solve all of the problems. If the matrix is singular, we must be more careful about the possibility of inconsistency. If the system is consistent, then we might as well pick an easy element of the nullspace when computing out a particular solution. (again, can choose things to get lots of zeros to make life easier.) This seems a strange way to do things given the ease of the first process, but it will have real application for systems of differential equations when we do variation of parameters.

Example of this last process We use the process just described to solve the system of equations

$$\begin{aligned} 7x+2y-3z &= 0 \\ x-y+2z &= 1 \\ 9x+9y-16z &= -5. \end{aligned}$$