211 LECTURE 18

Systems of Differential Equations, part I

1. Basic definitions and examples

A system of differential equations is just like an ordinary system of equations, except the constituent unknowns are functions and the equations are allowed to involve derivatives.

Examples

SIR model: This is a simple model for describing spread of an infectious disease. The simple derivation is in the text book. Assumptions: The disease is not fatal. A person who recovers is no longer susceptible to infection. Infection spreads between contact between individuals. Some other simplifying statements lead to the following set of equations for the number of Susceptible, Infected and Recovered people.

$$\begin{aligned} S' &= -aSI \\ I' &= \ \ aSI - bI \\ R' &= \qquad\quad +bI \end{aligned}$$

where a and b are some positive constants (representing 'reaction rates'). One can see that the third equation "uncouples" from the first two. It is not completely independent of I, S, but if we can solve the first two for I and S, then finding R is not a problem. Often, we'll see just the first two as the system to study.

Lorenz Equations: We've mentioned these before, I'll write them down again because they are so simple and important.

$$\begin{aligned} x' &= -ax + ay \\ y' &= \ \ rx \ - y - xz \\ z' &= \qquad\ -bz + xy \end{aligned}$$

Lotka-Volterra Model: This is a common model for systems of interacting populations with a predator (like foxes) and prey (like rabbits). If F is the number of foxes and R is the number of rabbits, then the model is something like

$$\begin{aligned} F' &= -aF + bRF \\ R' &= \ \ cR - dRF \end{aligned}$$

(Explain heuristics–lots of foxes bad for foxes, lots of rabbits good for rabbits, a rabbit and a fox meeting is bad for the rabbit and good for the fox.)

All three of these give first order,non-linear, autonomous systems of differential equations. Quickly recall the significance of each of these adjectives.

To compactify our notation and tap into multivariable calculus and linear algebra, one often writes systems in vector notation.

$$\mathbf{x}' = \mathbf{F}(\mathbf{x}, t).$$

Do one of the above this way.

2. Higher order equations and systems

We spent a lot of time on first order equations earlier in the semester. This is justified by what we do now. Any system of differential equations of any order in any number of unknowns can be rewritten as a first order system of differential equations. The key is to introduce new variables. We'll discuss this through some examples:

Circuit equations: The current I in a simple LRC circuit is modelled by $LI''+RI'+C^{-1}I = f(t)$, here L is inductance, R is resistance, C is capacitance and $f(t)$ is the electromotive force (an external forcing –generalizing battery).

Simple harmonic oscillators: $x'' + a^2x = 0$. (a simple example of the above.)

van der Pol's equation: $x'' + a(x^2-1)x' + x = 0$. (this arises in the study of vacuum tubes) Note: different look than the one in the text.

Coupled oscillators: Take a pair of masses hooked in a chain via springs to a wall. Using Newton's laws:

$$x'' = \frac{-(k_1+k_2)}{m_1}x + \frac{k_2}{m_1}y$$
$$x'' = \frac{k_2}{m_2}x - \frac{(k_2+k_3)}{m_2}x.$$

This technique of reduction has a major advantage: The numerical techniques and the theory we discussed for first order equations in one variable carry over almost verbatim!

As for numerical techniques, we just need to do a vector interpretation of each operation. Euler's method works just as well as before, we just need to work on all the various components of our vector at the same time. All of the numerical techniques in Matlab are designed to handle first order systems. The routines `eul, rk2, rk4` are exceptions, but they are locally designed. One could implement them without any trouble. . .

The Existence Theorem Let $R = (a, b) \times (\mathbf{c}, \mathbf{d})$ be a rectangle in the $t\mathbf{x}$-space. Suppose that the function $\mathbf{f}(t, \mathbf{x})$ is defined and continuous in all of R. Then for any point $(t_0, \mathbf{x}_0)$ in R, the initial value problem

$$\mathbf{x}'(t) = \mathbf{f}(t, \mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

has a solution $\mathbf{x}(t)$ defined on some interval containing t_0. Furthermore, the solution will be defined at least until the solution curve $t \mapsto (t, \mathbf{x}(t))$ leaves the rectangle R.

The Uniqueness Theorem Let $R = (a, b) \times (\mathbf{c}, \mathbf{d})$ be a rectangle in the $t\mathbf{x}$-space. Suppose that the function $\mathbf{f}(t, \mathbf{x})$ and its 'partial derivative' $\dfrac{\partial \mathbf{f}}{\partial \mathbf{x}}$ are defined and continuous in all of R. Suppose that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are two solutions to the initial value problem

$$\mathbf{x}'(t) = \mathbf{f}(t, \mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

Then $\mathbf{x}_1(t) = \mathbf{x}_2(t)$ on some small t-interval containing t_0.

Theorem (Continuous dependence on initial conditions) Suppose that $\mathbf{f}(t, \mathbf{x})$ and its derivative $\partial\mathbf{f}/\partial\mathbf{x}$ are both continuous in the rectangle R. Fix a "final time" $t_1 \in (a, b)$ for the initial value problem $\mathbf{x}' = \mathbf{f}(t, \mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}_0$. Let $\mathbf{x}_{\mathbf{x}_0}(t)$ be the solution of the initial value problem. Then $\mathbf{x}_{\mathbf{x}_0}(t_1)$ is a continuous function of $\mathbf{x}_0$.

3. Geometry and visualization

There are several good methods for visualization of the situation for higher dimensional first order systems.

- Plot individual components.
- Parametric curves in phase space. Especially useful for two dimensional autonomous equations! (hence phase plane), "vector fields" in phase plane,
- direction fields in three space.

  Discuss trying to see a non-autonomous equation in a 3d plot.
- Composite graph. all three of above together. time goes up. see individual components on vertical planes, phase plane on bottom, direction field and solution curve in the 3d box.
- in higher dimensions, we need to use different projections intelligently to figure things out. We can always do the individual components, but often we learn things by plotting various components against each other.

Discuss basic use of `pplane` to do the visualization. It will draw lots of pictures including a composite graph.

I recommended the books: *Nonlinear Dynamics and Chaos* and *Sync* both by Steven Strogatz and *Differential Equations, Dynamical Systems and and Introduction to Chaos* by Hirsch, Smale and Devaney.