211 LECTURES 20 AND 21

Linear Homogeneous Equations with constant coefficients, part I

1. Overview

Now the idea is to study constant coefficient systems. These provide us with a small class of equations that we can solve exactly. More importantly, we shall see later that they are more useful that just that.

Our starting point is the observation that single linear homogeneous equation has an exponential solution.

$$y' = ay \quad \Rightarrow \quad y(t) = Ce^{at}$$

Now, suppose that we are lucky enough to have the following situation: There is a non-zero vector $\mathbf{v}$ and a real number λ such that $A\mathbf{v} = \lambda\mathbf{v}$. Guided by the above, we guess that there is a solution of the form

$$\mathbf{y}(t) = e^{\lambda t}\mathbf{v}.$$

A quick check shows that this works! So now we have a simple thing to try.

Give definitions of eigenvalue, eigenvector. Note the relationship to the matrix equation $A - \lambda \cdot I$: eigenvalues are numbers such that this equation has a non-zero nullspace (i.e. the matrix is not invertible). The deal about non-invertibility of $A - \lambda I$ yields the characterization that eigenvalues are the roots of the characteristic polynomial of A, which is, by definition

$$p_A(\lambda) = \det(A - \lambda I).$$

Note that this allows us to compute eigenvalues by factoring!

Given an eigenvalue, λ, the corresponding eigenvectors are elements of the nullspace of $A - \lambda I$. So these form a subspace which can be determined by a row reduction computation.

2. Planar systems

We take up the specifics of this approach for planar autonomous homogeneous linear systems with constant coefficients. We shall look at three examples which show off the possible behaviors. In fact, these are specific examples, but they are completely representative. We shall state the general solution in each case and discuss the resulting phase plane pictures.

Example (distinct real eigenvalues) Use $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

Example (complex eigenvalues) Use $B = \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix}$.

Example (a repeated eigenvalue, first type) Use $C = \begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix}$.

Example (a repeated eigenvalue, second type) Use $D = \begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}$.

We discuss the solution of these test cases in great detail...

Is there a simple way to see which situation we will be in without actually doing all the computations? Well, sort of. In two dimensions our system $\mathbf{y} = A\mathbf{y}$ will have a characteristic polynomial of the form (do the computation)

$$p_A(t) = t^2 - T \cdot t + D,$$

where T is the trace of A and D is the determinant of A. By the quadratic equation, we can find the roots of this to be

$$\lambda_{1,2} = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

Thus we can set up a picture of which situation arises as follows: If $T^2 - 4D < 0$ then we have a pair of complex conjugate roots. If $T^2 - 4D > 0$ we have a pair of distinct real roots. If $T^2 - 4D = 0$ then we have a repeated root.