211 LECTURE 22

Is there a simple way to see which situation we will be in without actually doing all the computations? Well, sort of. In two dimensions our system $\mathbf{y} = A\mathbf{y}$ will have a characteristic polynomial of the form (do the computation)

$$p_A(t) = t^2 - T \cdot t + D,$$

where T is the trace of A and D is the determinant of A. By the quadratic equation, we can find the roots of this to be

$$\lambda_{1,2} = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

Thus we can set up a picture of which situation arises as follows: If $T^2 - 4D < 0$ then we have a pair of complex conjugate roots. If $T^2 - 4D > 0$ we have a pair of distinct real roots. If $T^2 - 4D = 0$ then we have a repeated root.

Go on to label the picture in the trace determinant plane with the type of behavior of the equilibrium point at the origin. Include spirals, centers, saddles, nodes, direction of rotation information, sink/source information.

Notice the possibilities for the behavior near the origin if λ_1 and λ_2 are the eigenvalues:

- If the eigenvalues are real:
  - **A saddle:** $\lambda_1 \cdot \lambda_2 < 0$
  - **A node:** $\lambda_1 \cdot \lambda_2 > 0$. If the signs are both positive, this is a source, if both negative, a sink.
  - **degenerate nodes and stars:** Only one eigenvalue $\lambda \neq 0$. If λ has geometric multiplicity one, we get a degenerate node. If λ has geometric multiplicity two, we get a star.
  - **Degenerate saddle:** One of λ_1, λ_2 is zero, the other isn't.
  - **A shear:** Zero is an eigenvalue of algebraic multiplicity two but geometric multiplicity one.
  - **Stationary system:** Zero is an eigenvalue of algebraic and geometric multiplicity two.
- if the eigenvalues come in a complex conjugate pair $\lambda, \overline{\lambda}$:
  - **Spiral Sink:** $\Re(\lambda) < 0$.
  - **Spiral Source:** $\Re(\lambda) > 0$.
  - **Center:** $\Re(\lambda) = 0$.

Where $\Re$ denotes "real part."

When we move on to higher dimensional systems, we will need to take care to understand how this set of possibilities changes. The fact is that several of them can happen together in the same system, and we will need a way to sort it all out.

Notice that in the last few lectures we had the following tricks work out:

- if there were complex conjugate eigenvalues, one could find complex solutions and the real solutions we built were just the real and imaginary parts of these solutions.
- In the case of repeated eigenvalues, we sometimes needed a weird trick of looking for vectors in the nullspace of $A - \lambda I$.

The second trick is slightly illuminated by the notions of "generalized eigenvalues", "algebraic multiplicity" and "geometric multiplicity".

Mention: Jordan Canonical Form and the cases it gives us. This should help in deciphering the troubles of higher dimensional systems.

Next time we'll introduce a new function to help sort all of this stuff out: The matrix exponential function.