211 LECTURE 23

The matrix exponential function

In a sense, we start over. Recall that the solution to a linear homogeneous equation in one variable

$$x' = ax, \quad \text{and} \quad x(0) = C,$$

is just $x(t) = e^{at}C$. I've written it backwards from how we usually do, but that's not a big deal. So, what we want to say is that the solution to the problem

$$\mathbf{x}' = A\mathbf{x}, \quad \text{and} \quad \mathbf{x}(0) = \mathbf{v},$$

is just $\mathbf{x}(t) = e^{tA}\mathbf{v}$. But what does it mean to take the exponential of a matrix?

Recall that for a number,

$$e^x = 1 + x + \frac{1}{2!}x^2 + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

So we try the following: **Definition:** The exponential e^A of a matrix A is the matrix

$$e^A = I + A + \frac{1}{2!}A^2 + \cdots = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

The tricky part here is to show that this makes any sense. That is, one has to show that the infinite sum always converges. We'll take this for granted in our class.
Example It is not difficult to use the definition to show that

$$\text{if} \quad A = \begin{pmatrix} \lambda_1 & 0 & \dots \\ 0 & \lambda_2 & \dots \\ 0 & \dots & \lambda_n \end{pmatrix}, \quad \text{then} \quad e^A = \begin{pmatrix} e^{\lambda_1} & 0 & \dots \\ 0 & e^{\lambda_2} & \dots \\ 0 & \dots & e^{\lambda_n} \end{pmatrix}.$$

So we readily see that in the 1×1 case (numbers), we get the old function back.
Example One can use the definition to show that

$$e^{\begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

But it is much less clear what happens when one tries to compute something like the exponential of $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$.

Important Properties of the matrix exponential

- $e^{\mathbf{0}} = I$,
- $A \cdot e^A = e^A \cdot A$,
- If A, B commute, that is if $AB = BA$, then $e^{A+B} = e^A e^B$,
- e^A is always invertible and $(e^A)^{-1} = e^{-A}$,
- If $\lambda, \mathbf{v}$ is an eigenvalue, eigenvector pair for A, then $e^{tA}\mathbf{v} = e^{\lambda t}\mathbf{v}$,
- $\dfrac{d}{dt}e^{tA} = Ae^{tA}$,
- $\mathbf{x}(t) = e^{tA}\mathbf{v}$ is a solution to $\mathbf{x}' = A\mathbf{x}, \mathbf{x}(0) = \mathbf{v}$.

Notice that the last few properties tell us that exponentials help us solve differential equations, and that if we can find eigenvectors, the solutions we get are just like the ones we found before. This is good news. What we need now is a good method for computing e^A. The full version of this will have to wait a few days, but we can do *something* for the unsolved case above. The idea is to use generalized eigenvectors effectively with the third property above and the definition.

Example Consider the case of $A = \begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}$. We have seen that $\mathbf{v} = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$ is a generalized eigenvector for A. So we can try to find a solution to our differential equation $x' = Ax, x(0) = \mathbf{v}$ as follows. Write $tA = 5tI + t(A - 5I)$ and note that $5tI$ commutes with $t(A - 5I)$. Thus our solution is is

$$\begin{aligned} x(t) &= e^{tA}\mathbf{v} = e^{5tI + t(A-5I)}\mathbf{v} \\ &= e^{5tI}e^{t(A-5I)}\mathbf{v} \\ &= e^{5t}(I + t(A - 5I) + \dots)\mathbf{v} \\ &= e^{5t}(\mathbf{v} + t(A - 5I)\mathbf{v} + \frac{t^2}{2!}(A - 5I)^2\mathbf{v} + \dots) \end{aligned}$$

But $(A - 5I)^2\mathbf{v} = 0$ because this is the level at which $\mathbf{v}$ becomes a generalized eigenvector! Thus the series vanishes after a finite number of terms and becomes very computable. If you check, this answer agrees with the one we got before.

This is the sort of thing we can always do. Use generalized eigenvectors and the series will terminate after a finite number of terms.

Example Try this out on the matrix

$$B = \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 1 \end{pmatrix}.$$

The key is that $(B - I)^2$ is zero.

General procedure for solving higher dimensional systems

Given an equation $x' = Ax$ we can solve the system by $x(t) = e^{tA}\mathbf{v}$ where $\mathbf{v}$ is a "general vector" of constants. In practice, this boils down to the following process (to be more precise about the solutions).

- Find the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A.
- For each eigenvalue λ_i
  - Find the algebraic multiplicity d,
  - Find the smallest integer k such that the nullspace of $(A - \lambda_i I)^k$ has dimension d,
  - Find a basis $v_1, \ldots v_d$ of this nullspace.
  - For $j = 1, \ldots d$ write out the (possibly complex-valued) solution

$$x_j(t) = e^{tA}v_j = e^{\lambda_i t}\left(v_j + t(A - \lambda_i I)v_j + \ldots + \frac{t^{k-1}}{(k-1)!}(A - \lambda_i I)v_j\right)$$

  - If $x_j(t)$ is complex valued, take its real and imaginary parts.
- Collect up all of the various x_j's from the different λ_i's. This will give a fundamental set of solutions.