211 LECTURE 24

Qualitative Properties of Higher dimensional systems

Things are significantly more complicated in higher dimensions. But we can say the following. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A.

- If $\mathbf{Re}(\lambda_i) < 0$ for each eigenvalue, then the origin is an asymptotically stable point (a "sink").
- If there is an eigenvalue λ_j such that $\mathbf{Re}(\lambda_j) > 0$, then the origin is an unstable equilibrium point.
- If all eigenvalues have $\mathbf{Re}(\lambda_j) > 0$, then we have a "source".
- There are higher dimensional versions of nodes, saddles, etc. They are defined in the "obvious ways".

Another look at higher order equations in one dimension

We have already discussed how to turn such an equation into a higher dimensional system. Recall that the first coordinate in such a system is where the function itself is kept, and the other coordinates are for its derivatives. In the case that the equation is linear like

$$y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = f(t), \tag{1}$$

the resulting system will be linear, too. It is not difficult to see that the words "homogeneous" and "constant coefficients" correspond, too. If we simply apply what we have learned about systems to this case and reinterpret it, we get a lot of consequences which are not immediately clear (and perhaps strange) from the original point of view.

So, consider the equation 1 together with the initial conditions $y(t_0) = y_0, y'(t_0) = y_1, \ldots y^{(n-1)}(t_0) = y_{n-1}$. Then we have the following results.

Theorem. Suppose that for all $t \in (\alpha, \beta)$, the functions $a_i(t)$ are continous. Then the problem has a unique solution which is defined for all the whole interval (α, β).

Theorem. If the equation is homogeneous, then the solution set has the form

$$\{c_1y_1(t) + \ldots c_ny_n(t) \mid c_i \in \mathbb{R}\}$$

where the collection of $y_i(t)$'s are a functionally independent set of solutions.

Theorem. The solutions $y_i(t)$ are functionally independent if and only if their *Wronskian* does not vanish some point. (The Wronskian is either never zero or always zero.)

Recall that the Wronskian is

$$W(t) = \det \begin{pmatrix} y_1(t) & \dots & y_n(t) \\ y_1'(t) & \dots & y_n'(t) \\ \dots & \dots & \dots \\ y_1^{(n-1)} & \dots & y_n^{(n-1)} \end{pmatrix}.$$

Theorem. If the coefficients are constant $a_i(t) = a_i$, then we can compute the characteristic polynomial of the matrix A corresponding to our problem as follows.

$$p_A(t) = \det(A - tI) = t^n + a_1 t^{n-1} + \dots + a_{n-1}t + a_n.$$

(Expand along the bottom row.)

We will still need to find a way to deal with inhomogeneous equations. This will happen next time.

Example: Consider a mass hanging from a spring under the influence of gravity. The spring has a "natural length". We use this to set the origin of a linear coordinate system which has positive up and negative down. We have a restoring force which by Hooke's law is given by $-kx$, $(k > 0)$, and gravity is given by $-mg$. Newton's laws give us

$$mx'' = F = -kx - mg.$$

This is linear but not homogeneous. We can change that by changing coordinates. The equilibrium point of this system is when $x' = 0 = x''$, that is at $x_0 = -mg/k$. We introduce a new coordinate $y = x - x_0 = x + mg/k$. Then $y' = x'$, $y'' = x''$ and our equation looks like

$$y'' = (-k/m)y.$$

So we set out new equations $x_1 = y$, $x_2 = y'$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T$. Then we get the system

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{pmatrix} \mathbf{x}.$$

To ease computation we introduce the number $\omega_0 = \sqrt{k/m}$ called the "natural frequency" of our oscillator. Applying our theory of linear systems, we see that the general solution of the system is

$$\mathbf{x}(t) = C_1 \begin{pmatrix} \cos(\omega_0 t) \\ -\omega_0 \sin(\omega_0 t) \end{pmatrix} + C_2 \begin{pmatrix} \sin(\omega_0 t) \\ \omega_0 \cos(\omega_0 t) \end{pmatrix}.$$

Of course, our desired function is the first coordinate of this, so we see that the general solution of the original equation is

$$x(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) - mg/k.$$