211 LECTURE 25

Today we study inhomogeneous equations $x' = Ax + f$. The solution set always has the form $x_p + x_g$ where x_p is a particular solution and x_g is the general solution to the associated homogeneous problem.

As usual, this means that the key to the inhomogeneous equation is to start by solving the homogeneous version. Suppose that $y_1(t), \dots y_n(t)$ is a fundamental set of solutions for the homogeneous equation $x' = Ax$. We form the matrix

$$Y(t) = [y_1(t), \dots, y_n(t)]$$

called a fundamental matrix for $x' = Ax$.

Notice that $Y' = AY$. (direct computation). Thus we have.

Theorem. A matrix valued function $Y(t)$ is a fundamental matrix for $x' = Ax$ if and only if

- $Y' = AY$, and
- $Y(t_0)$ is invertible for some (and hence any) t_0.

Fundamental matrices have two applications.

Application One: exponentials of matrices

Theorem. If $Y(t)$ is a fundamental matrix for the system $x' = Ax$, then the exponential e^{tA} of A can be computed as

$$e^{tA} = Y(t) \cdot Y(0)^{-1}.$$

The proof: Use the uniqueness theorem. Both e^{tA} and $Y(t)y(0)^{-1}$ are solutions to the (matrix) initial value problem $X'(t) = A \cdot X(t), X(0) = I$.

Example $A = \begin{pmatrix} -5 & 1 \\ -2 & -2 \end{pmatrix} \dots$

Application Two: Variation of Parameters

The general solution to $x' = Ax$ is given by $x(t) = e^{tA}v$, where v is a vector of constants. (Note: $Y(t) = e^{tA}$ is a fundamental matrix.) We will vary the constants v, and look for a solution of $x' = Ax + f$ in the form $x_p(t) = Y(t)v(t)$, where $v(t)$ is a vector of functions to be determined.

Since $x_p(t)$ is a solution, we see that

$$\begin{aligned} AY(t) + f(t) &= Ax_p(t) + f \\ &= x_p'(t) \\ &= Y'(t)v(t) + Y(t)v'(t) \\ &= AY(t)v(t) + Y(t)v'(t) \end{aligned}$$

For this to match, we need to have $Y(t)v'(t) = f(t)$. Of course, $Y(t)$ is invertible, so we need $v'(t) = Y(t)^{-1}f(t)$. Thus

$$v(t) = \int Y(t)^{-1} f(t)\ dt.$$

And therefore our particular solution must be

$$x_p(t) = Y(t) \cdot \int Y(t)^{-1} f(t)\ dt.$$

And we can then write out our general solution as

$$x(t) = Y(t)C + Y(t) \cdot \int Y(t)^{-1} f(t)\ dt,$$

for a vector of constants C, because $Y(t)C$ is the general solution of $x' = Ax$.
Example: We solve

$$x' = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix} x + \begin{pmatrix} e^t \\ e^t \end{pmatrix}$$

...do the computation...The answer is

$$y(t) = \begin{pmatrix} 2e^t + 9te^t \\ e^t + 6te^t \end{pmatrix} + C_1 \begin{pmatrix} 2e^{2t} \\ e^{2t} \end{pmatrix} + C_2 \begin{pmatrix} 3e^t \\ 2e^t \end{pmatrix}.$$