MATH 211 LECTURE 3

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1. Separable initial value problems

Discuss more efficient use of initial conditions in solution of separable first order equations. Use neglected example from last time: $y' = x^2y^2, y(0) = 1/2$ with solution $y = \dfrac{1}{2 - x^3/3}$.

2. Models of linear motion

For our first examples of applications, we use basic physics–the kind you've seen. "Motion in one dimension" Easiest to discuss things using Newton's laws: One of the laws of motion $F = ma$ and gravitation $F = GMm/r^2$. We make approximations for motion near surface of earth (a commonly observed phenomenon), get influence of gravity is $GM/r^2 = g = \text{const} \approx 9.8$ m/s. Note: instead of second order equation for the position, we get first order equations in the velocity–we can try these!

example toss of an object (say chalk). Get $m \cdot v' = -m \cdot g, v'(0) = 20$ m/s. Easy to solve.

example add other forces! most common is air resistance. The two types typical are proportional to velocity and velocity squared: add these equations to the list. $mv' = -mg - Av$ and $mv' = -mg - Av|v|$. solve these in special cases? Say add linear resistance to my equation above of the form $-.05v$.

3. Linear equations

An important class of equations that we can solve is the collection of *linear equations.* These are equations whose normal form looks like

$$y' = f(x)y + g(x).$$

It is not uncommon to see these out of normal form in the following manner:

$$f(x)y' + g(x)y = h(x)$$

In fact, we like the second version better. This general type where h is not identically zero is called an *inhomogeneous* equation. If $h(x) = 0$ identically, then the equation is said to be *homogeneous.* We can define an *associated homogeneous equation* by removing the h term.

3.1. **integrating factors.** This method is based on a way to use the product formula. The idea is to try and use it backwards! We are looking to rewrite the equation so that a product is present! This will simplify our integration problems.

The idea is that basically the equation had a factor common to both sides, and someone cancelled them before giving you the problem. Our goal is to put it back in to fix things.

Start with equation $y' + g(x)y = h(x)$. (Notice change in starting form.) We want to realize LHS as the derivative of a product like this:

$$(f(x)y)' = f(x)y' + f'(x)y = f(x)\left(y' + \frac{f'(x)}{f(x)}y\right).$$

This means that to realize our LHS as a product, we need to multiply the equation through by the function f which satisfies $f'/f = g$. This function is called an *integrating factor*. Then we can just integrate up and rearrange. Note that any one integrating factor will do, don't need the general solution.

example $y' + 6y = 1$. If u is to be an integrating factor, we want

$$(uy)' = u(y' + \frac{u'}{u}y) = u(y' + 6y)$$

Hence, we must solve $u'/u = 6$. This has a solution of $u = e^{6x}$. We multiply through by this function to see

$$e^{6x}y = \int (e^{6x}y)' = \int e^{6x}(y' + 6y) = \int e^{6x} = \frac{1}{6}e^{6x} + C$$

We then rearrange to find $y(x) = \frac{1}{6} + Ce^{-6x}$.

Note that an equation may have more than one integrating factor, but any one will do. Very common to have exponentials in the integrating factor.

example $y' - y = \cos(x)$. We want to find a function u so that $(uy)' = uy' + u'y$ is equal to $u(y' - y)$. Setting these equal, we find that u must satisfy $u' = -u$. This has solution $u = e^{-x}$. So we find that

$$e^{-x}y = \int (e^{-x}y)'\,dx = \int e^{-x}(y' - y)\,dx = \int e^{-x}\cos(x)\,dx = \frac{e^{-x}}{2}(\sin(x) - \cos(x)) + C.$$

We rearrange to find $y = \frac{\sin(x) - \cos(x)}{2} + Ce^{x}$.

example Students try $y + \frac{y}{x} = 8x^6$. Answer is $y(x) = x^7 + \frac{C}{x}$. The integrating factor is x.