MATH 211 LECTURE 4

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1. Linear Equations, continued

1.1. **Variation of parameters.** This is a different way to put the product formula to use. It is not so different in idea from the last method, but very different in implementation.

Solve homogeneous equation by separation of variables. $y = Ce^{\int g(x)/f(x)\ dx}$ is the answer.

Proposition: *If $y_p(t)$ is a particular solution to the linear first order equation $f(x)y' + g(x)y = h(x)$ and $y_H(x)$ is a solution to the associated homogeneous equation $f(x)y' + g(x)y = 0$, then $y = y_p + y_H$ is a solution to the inhomogeneous problem.*

give simple proof. So problem is now, how do we even find a particular solution to the inhomogeneous equation? Well, one way to fix the whole thing is with *variation of parameters.*

Basic idea is this: our solution to the homogeneous problem is supposed to be related, so we look for solutions of the form $y(t) = v(x)y_H(x)$. That is, we postulate that a solution looks like this, and we try to figure out what the function v has to be. This leads to the following setup:

If $y = v \cdot y_H$ is a solution, we must have

$$\begin{aligned} h(x) &= f(x)(v \cdot y_H)' + g(x)(v \cdot y_H) \\ &= f(x)(v' \cdot y_H + v \cdot y_H') + g(x)(v \cdot y_H) \\ &= v\,(f(x)y_H' + g(x)y_H) + f(x)v' \cdot y_H \\ &= 0 + f(x)v' \cdot y_H. \end{aligned}$$

This means that we can find v as a solution to the first order *separable* equation $v' = \dfrac{h(x)}{f(x)y_H}$, which is okay because we already have y_H at our disposal.

Note that if you try the method and you don't get a big cancellation, you have made a mistake–start over. It is also a good idea to check that your answer takes the form "particular solution"+"general solution to the homogeneous problem". i.e. $y_p + y_H$–this should happen, too, but it is not always obvious.

example $y' = y + \cos(x)$. The associated homogeneous equation is $y' = 2y$. This is separable with a particular solution of $y_H(x) = e^x$. We then look for a solution of the form $y(x) = v(x)y_H(x)$. As above, we end up at the equation $v' = \dfrac{\cos(x)}{e^x}$. We integrate this to find... $v(x) = \frac{e^{-x}}{2}(\sin(x) - \cos(x)) + C$. So the general solution looks like

$$y(x) = v(x) \cdot y_H(x) = \frac{\sin(x) - \cos(x)}{2} + Ce^x$$

example $y + \frac{y}{x} = 8x^6$. Answer is $y(x) = x^7 + \frac{C}{x}$ (In this case $y_H = 1/x$.)

example Students try $y' = y + x$. Final solution is: $y(x) = Ce^x - x - 1$.

2. Mixing Problems

Another application problem set. The mixing problems are a common way to set up some differential equation problems. Usual deal: assume that whatever is happening is mixed *instantaneously*. a bit unphysical, but gives us math we can do now, not next year.

These are really word problems. Two things worth remembering for setting up the math part properly:

- dx/dt = Rate IN - Rate OUT.
- Watch your physical units. They often tell you exactly what to do.

We cover examples that are problem 3 and 4 from section 2.5 of the text.

example(Text, ch2.5# 3) The integrating factor required for problem 3 is $u = e^{t/20}$. The resulting solution is $x = 25 - 23e^{-t/20}$. Note we don't actually need to find the constant to solve this problem. Now use dfield6 to show approximately the same thing.

example(Text, ch2.5# 4) This leads to a separable homogeneous linear equation. solve by separating variables to get $\ln|x(t)| = \ln(25) - At/500$. We want $x(60) \leq 5\ lb$. Hence, $2\ln(5) - 60A/500 \leq \ln(5)$ is what we need. In the end, this means that we need $A \geq \frac{25}{3}\ln 5 \approx 13.412\ gal/min$.