MATH 211 LECTURE 5

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1. primer on partial derivatives

For a while now, we have been dealing with functions of two variables, though it may not have been obvious. In order to keep track of what is happening for today, we need the concept of *partial derivative*. What it is: technically, you do a funny limit, just like the regular case, but you only care about one variable at a time. But for a functional definition (instead of a technical one), what you do is this: pretend the other variables are constants and take the derivative with respect to the indicated variable. In order to keep track of this stuff, and to reduce confusion with ordinary differentiation, we use a different symbol ∂ –a Cyrillic d.

example Find the partial derivative of $F(x, y) = x^2 + \cos(xy) - xy^3$ with respect to x, then with respect to y. Compare. (use notation $\partial F/\partial x$)

example (for students to try) Compute the first partial derivatives of $F(x, y) = \cos(x)\cos(y) - \sin(x)\sin(y)$. Answer is: $\partial F/\partial x = -\sin(x)\cos(y) - \cos(x)\sin(y)$ and $\partial F/\partial y = -\cos(x)\sin(y) - \sin(x)\cos(y)$.

Mention higher derivatives, and the important fact: Equality of mixed partials for smooth functions.

2. Differentials, implicit solutions, restatement of ODE

To put this multivariable calculus to use, we need to reinterpret our problem a bit. Often, when solving a differential equation, we get left with an "implicit" solution. That is, there is an equation involving x and y which relates the two (and therefore *implicitly defines y as a function of* x), but no good way to find an equivalent formula like $y(x) =$ some function of x.

How do we check that such an expression actually is a solution? We use implicit differentiation. Compare with the example from lecture 2, the equation $y' = \frac{e^{-y}}{1+y}$, $y(0) = 2$ was found to have solution defined by $ye^y = x + 2e^2$. We'll really check this now. First rearrange to the form $F(x, y) = ye^y - x - 2e^2$

Claim: the *level curves* $F(x, y) = C$ (where C is constant) define solutions to the differential equation above, and the one containing the point $(x, y) = (0, 2)$ is our curve. Do this using y' notation and chain rule, recalling $y = y(x)$.

Note that along the way, we found an equation of the form $\partial F/\partial x + \partial F/\partial y \cdot \frac{dy}{dx} = 0$. Note the relationship to the normal form of a first order ODE. Rewrite the implicit differentiation step to look like this. Next, "multiply through by dx."

This suggests the formalism of *total differentials*:

$$dF = \frac{\partial F}{\partial x}\ dx + \frac{\partial F}{\partial y}\ dy$$

and *differential form* version of a differential equation:

Restatement of first order differential equation problem.
A first order differential equation is a statement that a differential form is zero:

$$P(x, y)\ dx + Q(x, y)\ dy = 0.$$

When we try to solve such an equation, we look for a function $F(x, y)$ such that the solutions are described implicitly by the level sets $F(x, y) = C$.

example The total differential of $F(x, y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ is

$$dF = (-\sin(x)\cos(y) - \cos(x)\sin(y))\ \ dx + (-\cos(x)\sin(y) - \sin(x)\cos(y))\ \ dy.$$

3. Exact Equations

By our above discussion, one way to produce a function $F(x, y)$ whose level curves give solutions to the differential equation is to find a such a function with the following arrangement: $\partial F/\partial x = P(x, y)$ and $\partial F/\partial y = Q(x, y)$. That is, it is sufficient (but not necessary) that we find a function of two variables whose total differential is the given differential form. Then the level sets of our function define solution curves, often called *integral curves*, to the differential equation.

Defn A differential form is called *exact* if it is the total differential of a function. We also call the corresponding differential equation exact.

New problems:(1) How can one tell if a differential form (equation) is exact?
(2)If an equation is exact, how do we find the function F?

We can answer these a lot of the time, and we do both at once in the following theorem.

Theorem Suppose that $\omega = P\ dx + Q\ dy$ is a differential form where both P and Q are continuously differentiable functions. Then ω is exact if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Note: there is some funny business with connected/disconnected regions in the xy-plane, but this will almost never get in our way.

For a full proof, see text. We can explain how it goes like this. Suppose that ω is exact. Then the integral function F must be twice continuously differentiable. This means that mixed partials are equal, and we get the statement.

For the other direction, we actually give a method of finding F which is guaranteed to work if the condition is satisfied.

4. The method

Suppose that $\omega = P\ dx + Q\ dy$ is exact (in the sense that we know the condition of the theorem holds). Then, to find F do the following:

(1) integrate the equation $\partial F/\partial x = P$ with respect to x. (treat y as a constant in this step.) Write the arbitrary constant that comes out as a function of y alone. $F = \int P\ dx + C(y)$.

(2) differentiate the resulting expression with respect to y, and recall

$$Q = \frac{\partial F}{\partial y} = \frac{\partial}{\partial y}(\int P\ dx) + C'(y).$$

Compare with Q to find a differential equation which describes $C(y)$. Some major cancellation should happen here.

(3) Solve the equation to find $C(y)$.

(4) Double check your answer by implicit differentiation.

example $\omega = (x+y)\ dx + (x-y)\ dy = 0$. (equivalent to $dy/dx = (x+y)/(x-y)$.) Solution is $F(x,y) = x^2/2 + xy - y^2/2 = C$.

example $(1 + y/x)\ dx - (1/x)\ dy = 0$. Not exact. Send help!

example $(y/x - 2)\ dx + \ln(x)\ dy = 0$. Solution is

$$F(x,y) = y\ln(x) + 2x = C.$$