MATH 211 LECTURE 6

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The Theory of Differential Equations, part I

We abandon our study of how to functionally solve differential equations to discuss the important theory of differential equations, at least in the context of first order equations in one variable.

1. The existence theorem

A fundamental question for us is the following:

When can we be sure that a differential equation has a solution?

Actually, it is better to consider the question of when an initial value problem has a solution, as things can depend on the kind of initial condition you choose. An answer is:

The Existence Theorem Let $R = (a, b) \times (c, d)$ be a rectangle in the tx-plane. Suppose that the function $f(t, x)$ is defined and continuous in all of R. Then for any point (t_0, x_0) in R, the initial value problem

$$x'(t) = f(t, x), \quad x(t_0) = x_0$$

has a solution $x(t)$ defined on some interval containing t_0. Furthermore, the solution will be defined at least until the solution curve $t \mapsto (t, x(t))$ leaves the rectangle R.

The geometric picture to have in mind is exactly the kind which is described by a direction field. The solution curves are exactly the things which are plotted by `dfield`.

A couple of remarks are in order:

(1) The theorem is stated for initial value problems which have the differential equation in *normal form*. To apply the theorem, we need to transform our equation into normal form.
(2) The theorem is not very specific about how long a solution exists. That is, how big of an interval of t for which the solution exists is not very explicit. Is there any situation in which we can expect better?
(3) The defining function f is required to be continuous. Sometimes, we care about a situation in which this doesn't hold. What happens?

Some examples with pictures made by dfield. Use window $t \in (-2, 2)$, $x \in (-3, 3)$.

example (not normal form–hence no solution. Still no contradiction.) $tx' = t^2 - 1, \quad x(0) = 2$.

example (interval of existence is restricted by leaving box–discontinuity of f.) $x' = \dfrac{2}{(x+2)(t^2-1)}, \quad x(0) = 0$.

example (interval of existence restricted even thought everything is nice.) $x' = 1 + x^2, \quad x(0) = 0$.

example (how to paste solutions for discontinuous f, when x remains bounded.) See example in the text.

2. Picard's method of iteration

Here is an idea of the first general proof of the existence theorem. This proof is due to Emile Picard–read bio from Simmons' footnote.

The idea is to successively approximate the solution to the differential equation and show that we must converge to an actual solution.

First, if $x(t)$ is our solution, then it satisfies $x'(t) = f(t, x(t))$. So, if we integrate with respect to t, we see that $x(t)$ is supposed to be a solution to the equation

$$x(t) = x_0 + \int_0^t f(s, x(s))\ ds.$$

This is the relation we will use over and over again.

Our first approximation is that $x_0(t) = x_0$ is constant. Then taking this as known, we define the next approximation by

$$x_1(t) = x_0 + \int_0^t f(s, x_0)\ ds.$$

We repeat this over and over! If we have the nth approximation $x_n(t)$, we define the next one by

$$x_{n+1}(t) = x_0 + \int_0^t f(s, x_n(s))\ ds.$$

The hard part is now to show that the series *of functions* $x_n(t)$ converges to a solution as $n \to \infty$.

example We can easily solve the initial value problem $x' = x + t, x(0) = 1$ by variation of parameters, or by choosing an integrating factor, as the equation is linear. The solution is $x(t) = 2e^t - t - 1$. (check this!). What does Picard's method give us?

The successive approximates are:

$$x_0(t) = 1$$
$$x_1(t) = 1 + \int_0^t (1+s)\,ds = 1 + t + \frac{t^2}{2!}$$
$$x_2(t) = 1 + \int_0^t (1+s+\frac{s^2}{2!}+s)\,ds = 1 + t + t^2 + \frac{t^3}{3!}$$
$$x_3(t) = 1 + \int_0^t (1+s+s^2+\frac{s^3}{3!}+s)\,ds = 1 + t + t^2 + \frac{t^3}{3} + \frac{t^4}{4!}$$
$$x_4(t) = 1 + \int_0^t (1+s+s^2+\frac{s^3}{3}+\frac{s^4}{4!}+s)\,ds = 1 + t + t^2 + \frac{t^3}{3} + \frac{t^4}{3\cdot 4} + \frac{t^5}{5!}$$
$$\vdots \quad = \quad \vdots$$
$$x_n(t) = 1 + \int_0^t (x_{n-1}(s)+s)\,ds = 1 + t + 2\left(\frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^n}{n!}\right) + \frac{x^{n+1}}{(n+1)!}$$

A little thought shows that this converges to $x(t) = 1+t+2(e^t-t-1)+0 = 2e^t-t-1$. Yeah!

3. The interval of existence for linear equations

By a careful examination of the estimates in the proof, one can show that things are a bit better for linear equations. The result is as follows.

Theorem (Global existence for linear equations) Consider the initial value problem

$$x'(t) = P(t)x + Q(t), \quad x(0) = x_0$$

for functions P and Q which are continuous on the interval $t \in (a, b)$. This has a solution which exists for all t in the interval (a, b).

What happens is that the equation is good in a 'vertically infinite' strip, and we can show that the solution grows no more than exponentially fast. (i.e. there is a bound on the size of the derivative!) Thus, it can only leave the strip by exiting the sides!

example Compare with the case of tan x.

example $x' = x + t$, draw boxes with big verticals....