MATH 211 LECTURE 7

TJ HITCHMAN

The Theory of Differential Equations, Part II

We continue our study of the theory of differential equations. Today we focus on two questions:

Is it possible to have more than one solution to an initial value problem?

What happens if we change the initial conditions a little? Are the solutions still close?

These seem at first like the perverse questions of a mathematician, but they have importance for scientific applications. Discuss 'uniqueness of history' and 'experimental/measurement error'.

1. The Mean value theorem and a fundamental estimate

We start by recalling an important fact from calculus.

The Mean Value Theorem Let f be a function which is defined and has a continuous derivative on some interval containing (a, b). Then there is some point c lying between a and b such that

$$f(b) - f(a) = f'(c) \cdot (b - a). \tag{1}$$

Now we can get down to business.

Lemma (The basic estimate) Let $R = (a, b) \times (c, d)$ be a rectangle in the tx-plane. Suppose that $f(t, x)$ and its partial derivative $\dfrac{\partial f}{\partial x}$ are defined and continuous in all of R, also that $\dfrac{\partial f}{\partial x}$ is bounded on R with

$$M = \max_{(t,x)\in R} \left| \frac{\partial f}{\partial x} \right|.$$

Pick a pair of points (t_0, x_0) and (t_0, y_0) lying on the same vertical line in R, and let $x(t)$ and $y(t)$ be the solutions which correspond by the existence theorem. Then as long as both solution curves $(t, x(t))$ and $(t, y(t))$ lie inside R, we have

$$|x(t) - y(t)| \le |x(0) - y(0)| \cdot e^{M|t-t_0|}. \tag{2}$$

Give the argument using (1). Idea is to study the function $G(t) = y(t) - x(t)$, applying the mean value theorem to its derivative, then integrating to get the estimate.

2. The Uniqueness theorem

Now we have the tool we need to discuss the question of uniqueness or non-uniqueness of solutions to an initial value problem.

2.1. **The theorem. Theorem** (The Uniqueness Theorem) Let $R = (a, b) \times (c, d)$ be a rectangle in the tx-plane. Suppose that the function $f(t, x)$ and its partial derivative $\dfrac{\partial f}{\partial x}$ are defined and continuous in all of R. Suppose that $x_1(t)$ and $x_2(t)$ are two solutions to the initial value problem

$$x'(t) = f(t, x), \quad x(t_0) = x_0.$$

Then $x_1(t) = x_2(t)$ on some small t-interval containing t_0.

The proof is as follows: If $x_1(t)$ and $x_2(t)$ are both solutions, we can apply our basic estimate 2 to see that $|x_1(t) - x_2(t)| \leq 0$. That is, x_1 and x_2 must agree as functions of t.

2.2. **The geometry of uniqueness/non-uniqueness.** We go back to the direction field picture. What we see is that in the case of the uniqueness theorem, no more than one solution curve can pass through any point in the rectangle R.

example (of non-uniqueness) See problem 24 of ch 2.7 in text. The equation $x' = -\sqrt{x}$ does not have continuous derivative along the line $x = 0$. This leads to lots of solutions through $(0, 0)$.

3. Dependence on initial conditions

We can use our basic estimate just a bit more cleverly to get the following statement. **Theorem** (Continuous dependence on initial conditions) Suppose that $f(t, x)$ and its derivative $\partial f/\partial x$ are both continuous in the rectangle R. Fix a "final time" $t_1 \in (a, b)$ for the initial value problem $x' = f(t, x), \quad x(t_0) = x_0$. Let $x_{x_0}(t)$ be the solution of the initial value problem. Then $x_{x_0}(t_1)$ is a continuous function of x_0.

Draw the picture. Discuss how this follows from the basic estimate (2).

Just because things are continuous doesn't imply that we are out of the woods. Things can still be difficult because the estimate 2 can get worse *exponentially fast* as t grows. We only know that solutions are close for some small time interval, and if the constant M is big, this interval may be very small. This leads to the study of the sensitivity of dependence on initial conditions–kind of like studying the "slope" of a continuous function. We get the term *chaos* out of this study.

example Consider the equation $x' = 3x$. This has solutions $x(t) = 0$ and $y(t) = .01e^{3t}$. These diverge pretty quickly after $t = 2$ as $y(2) = .01e^6 \approx 4.034$.