MATH 211 LECTURE 8

Qualitative Analysis of first order autonomous equations

In this lecture we begin study of the more modern part of differential equations: *qualitative analysis*. We shall usually forgo a detailed solution except as verification of our work, and instead try to divine features of the solution curves directly from the form of the differential equation.

1. Autonomous equations and a new look at direction fields

A first order differential equation is said to be *autonomous* when it is in normal form and the defining function has no dependence on the independent variable. That is, when it is in the form of

$$x' = f(x).$$

The important point being that the right hand side doesn't depend on t. This happens often enough (indeed, we will learn how to arrange it) that it deserves its own name and its own day of study.

example $y' = \sin(y) \cdot y^2$ is autonomous

example $y' = t\sin(y)$ is not.

As long as it is clear what the independent variable is supposed to be, it is easy to determine if an equation is autonomous.

Notice that in making a direction field, we don't need any information about t to compute our little slope lines! We see that direction fields for autonomous equations are translation invariant in the horizontal direction.

Consider our examples of a point moving in a line: Somehow, this corresponds to the situation described by "the rules of motion don't change in time, only in space."

This simpler form for the direction field makes things a bit easier to think about. It is the key insight for what we will do today.

2. Equilibrium points and solutions

A system is often said to be *in equilibrium* if it doesn't change in time. We adopt this idea and say that a solution is an *equilibrium solution* of our differential equation if it is a constant function.

example The equilibrium solutions of $x' = 1 - x^2$ are $x(t) = \pm 1$.

In general, this is exactly what happens. An equilibrium solution is a solution of the form $x(t) = x_0$ where x_0 is a number that makes $(x' =)\ f(x_0) = 0$. Often, the point x_0 is called an *equilibrium point*.

example Continue the above with dfield to see the equilibrium solutions. Then do $y' = (y+1)(y^2-9)$ "together".

3. THE PHASE LINE–A GRAPHICAL TECHNIQUE

Since the direction field is really a set-up for viewing 2-dim information, and we are precisely in the situation where the relevant info is 1-dim, one might suspect there is a simpler graphical representation of the information related to an autonomous equation. This is so. We use a *phase line*.

Discuss how the graph of f gives us a phase line. Discuss the relationship to the direction field. Be sure to point out the change in axis, too. Finally, point out that dfield has a phase line option in it, and if you slow the computation down, you can even see the little bead trace the relevant path up or down the line.

example Do example for $x' = 1 - x^2$.

example $P' = 10P(1 - \frac{P}{200})$.

4. STABILITY, IN WORDS AND PICTURES

Discuss how easy it is to see "long term" behavior from the phase line. But we need to clarify how things work.

An equilibrium point x_0 is called *stable* if points which start nearby stay nearby. It is called *asymptotically stable* if a point which starts nearby must eventually limit onto x_0. It is called *unstable* if nearby points fly away. It is called *semi-stable* or *half-stable* if it looks stable on one side and unstable on another. We won't see many examples of points which are stable but not asymptotically stable. Note that here asymptotically stable does imply stable. This can fail in higher dimensions. Discuss these definitions with examples and the relevant pictures of phase lines, direction fields. Show off the solid/open dots pictures.

example Do $x' = 1 - x^2$. an asymp stable and an unstable point.

example $y' = (y+1)^2$ a half stable point.

example $y' = 0$ has only stable points, but none are asymptotically stable.

5. THE ANALYTICAL TEST FOR STABILITY, AND ITS PICTURE PROOF

It is actually pretty easy to get a good test for the forms of stability. We get the one from the text without any trouble

Theorem (first derivative test for stability) Let x_0 be an equilibrium point for the autonomous equation $x' = f(x)$. Suppose that f is continuously differentiable.

- If $f'(x_0) < 0$, then x_0 is an asymptotically stable equilibrium.
- If $f'(x_0) > 0$, then x_0 is an unstable equilibrium.
- If $f'(x_0) = 0$, then the test fails.

Discuss the picture proof. Go on to draw representative cases in terms of graphs and phase lines.