Math 211, Spring 2005
Exam II
Solutions

Problem 1 Euler's method is an algorithm for approximating solutions to initial value problems numerically. Suppose we have an initial value problem of the form

$$\begin{aligned} x' &= f(x,t) \\ x(t_0) &= x_0. \end{aligned}$$

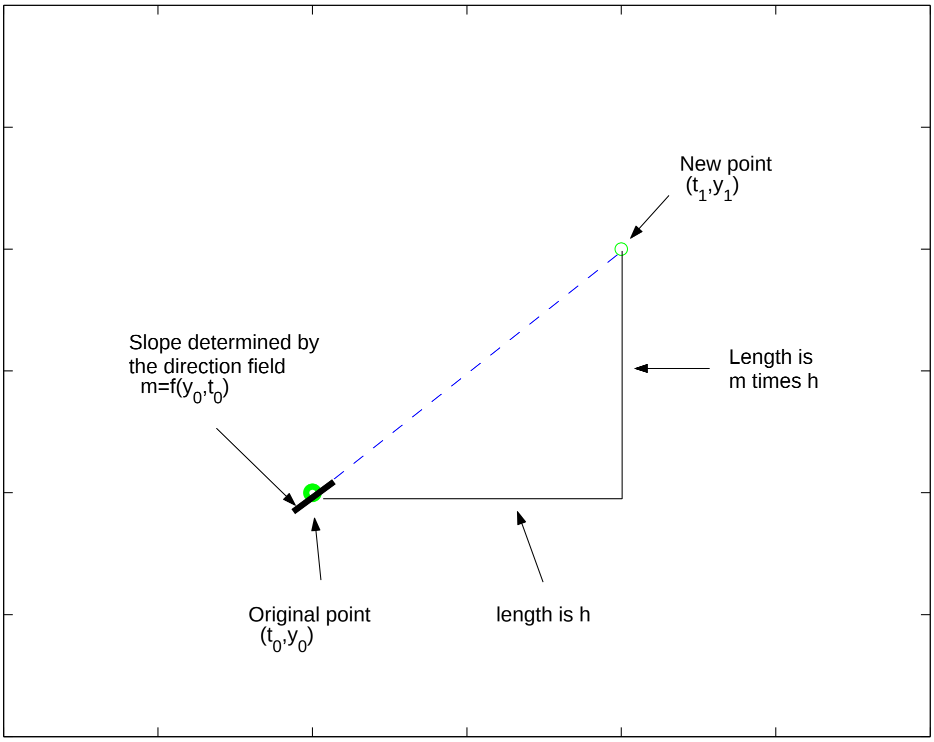
Then Euler's method is the inductive algorithm defined as below.

(1) Fix a step size $h > 0$.
(2) Set $t_0 = t_0$ and $x_0 = x_0$ (i.e. begin at initial point from the problem).
(3) If t_i, x_i are defined, then set

$$\begin{aligned} t_{i+1} &= t_i + h \\ x_{i+1} &= x_i + h \cdot f(x_i, t_i). \end{aligned}$$

(4) repeat step (3) until we reach the end of our interval.

Each individual step as in (3) can be visualized as the following linear approximation scheme.



This method is nice because it is simple and easy to implement on a computer. It also requires only one functional evaluation per step, which saves on computing time.

We can also bound the cumulative error in this procedure as

$$\text{total error} < C \cdot h,$$

where C is a constant that depends on the function f and the length of the interval in question. This means that to reduce the error by a factor of 2, we must cut the step size in half, thereby doubling the number of computations required.

Problem 2 $A \cdot B^T = \begin{pmatrix} 2 & 8 \\ 4 & 18 \end{pmatrix}$.

Problem 3 The nullspace is

$$\mathcal{N}(C) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} = t \cdot \begin{pmatrix} -3 \\ 2 \end{pmatrix} \mid t \in \mathbb{R} \right\}.$$

Problem 4 The row echelon form of this matrix is

$$D = \begin{pmatrix} 1 & 3 & 5 \\ 0 & -17 & -34 \\ 0 & 0 & 1 \end{pmatrix},$$

so $\det(D) = -17$. For the determinant of E, we expand along the second row to see

$$\det(E) = -0 \cdot \det(\text{stuff}) + 0 \cdot \det(\text{stuff}) - 2 \cdot \begin{pmatrix} 6 & 5 & 3 \\ 5 & 5 & 3 \\ 1 & 0 & 0 \end{pmatrix} + 1 \cdot \det \begin{pmatrix} 6 & 5 & 0 \\ 5 & 5 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

The first non-zero term here is actually zero, too, because in the 3×3 matrix involved the first row is the sum of the first two. So, we expand the last term along the bottom row to see

$$\det(E) = 1 \cdot \det \begin{pmatrix} 5 & 0 \\ 5 & 1 \end{pmatrix} - 0 \cdot \det(\text{stuff}) + 1 \cdot \det \begin{pmatrix} 6 & 5 \\ 5 & 5 \end{pmatrix} = 5 + (30 - 25) = 10.$$

Problem 5 This is not possible. If $\det(A) \neq 0$, then A is invertible. Hence any equation $Ax = b$ has a solution of $x = A^{-1}b$. This means that the system must be consistent.

Problem 6 This is possible. There are several ways to write an example, but the easiest one I can think of is the system

$$\begin{aligned} x \quad &= 0 \\ y &= 0 \\ x + y &= 1. \end{aligned}$$

The matrix form of this has matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ which has trivial nullspace.

Problem 7 The set of solutions is

$$\mathcal{S} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \mid t \in \mathbb{R} \right\}.$$

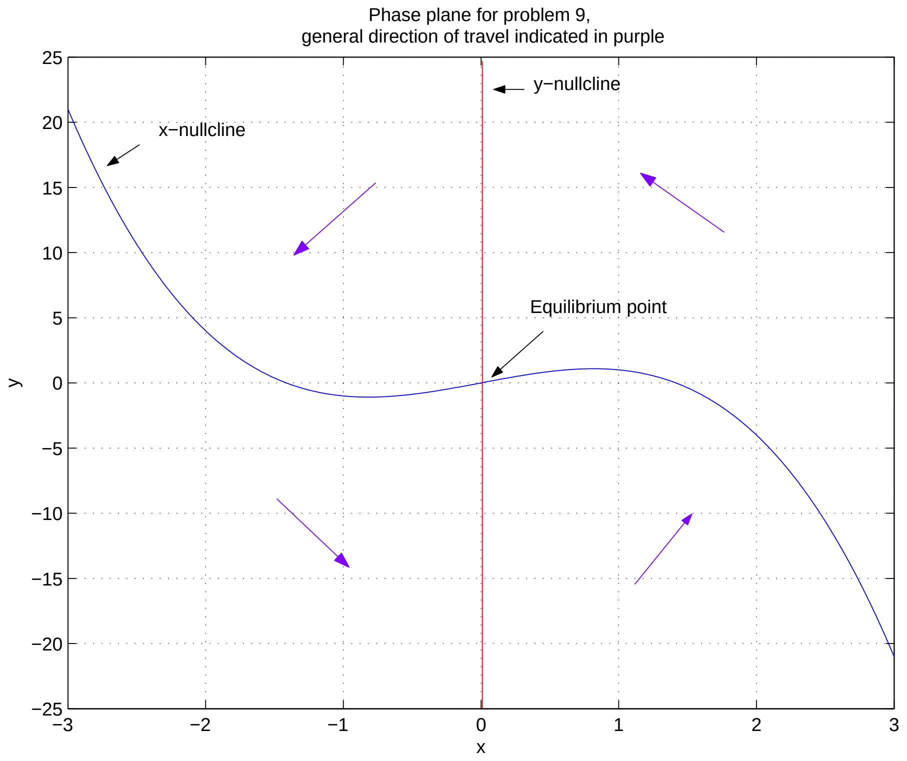
Problem 8 We introduce new variables $y = x', z = x'' = y'$. Then the equivalent system is

$$\begin{aligned} x' &= y \\ y' &= z \\ z' &= \cos(t)z - \sin(t)y + e^t x + \tan(t). \end{aligned}$$

This has matrix form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ e^t & -\sin(t) & \cos(t) \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \tan(t) \end{pmatrix}$$

Problem 9 The x-nullcline is the curve $y = 2x - x^3 = x(2 - x^2)$. Note that y' is negative above this curve (leftward travel), and positive below the curve (rightward travel). They y nullcline is the y-axis $x = 0$. To the right of this curve, y' is positive (upward travel), to the left y' is negative (downward travel). The only place we get an intersection of these curves to determine an equilibrium point is the origin $x = 0, y = 0$. Therefore our rough picture looks like the following.



Phase plane for problem 9,
general direction of travel indicated in purple
y−nullcline
x−nullcline
Equilibrium point
y
x
25
20
15
10
5
0
−5
−10
−15
−20
−25
−3
−2
−1
0
1
2
3