Math 212 Spring 2006
Exam #1: Solutions

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1. Let $P = (2,3,1), Q = (2,2,2), R = (3,3,-1)$.

   (a) Find the equation of the plane through P, Q and R.
   Solution: We see that the plane is parallel to the vectors $v = Q - P = (0,-1,1)$ and $w = R - P = (1,0,-2)$ hence the normal direction to the plane is $v \times w = (2,1,1)$. Therefore, we conclude that the plane is the set of points $X = (x,y,z)$ which satisfy

   $$0 = (X - P)\cdot(v \times w) = 2(x-2) + (y-3) + (z-1) = 2x + y + z - 8.$$

   (b) Let l be the line given by $x = 2 - t, y = 3t, z = 1 + 2t$. Find the intersection point of the plane and the line.
   Solution: a point of intersection must lie on both the plane and the line, and thus satisfies the equations given and the equation from (a). We substitute to obtain:

   $$0 = 2(2-t) + 3t + 1 + 2t - 8 = 3t - 3.$$

   Thus we must have $t = 1$. This corresponds to the point $(1,3,3)$.

2. Let P be the plane given by the equation $x_1 - 2x_2 + 3x_3 + 4 = 0$ and let Q be the plane given by all points of the form

   $$(2,4,1) + \lambda(3,3,1) + \mu(2,-1,1), \qquad \lambda, \mu \in \mathbb{R}.$$

   Determine whether P and Q are parallel or not.

   Note: Two planes are called parallel if they do not intersect, an equivalent condition is that two planes are parallel if their normal vectors are parallel.

   Solution: A vector normal to P can be found by thinking of P as the 0-level set for the function $f(x_1,x_2,x_3) = x_1 - 2x_2 + 3x_3 + 4$ and taking the gradient at some convenient point. This is $n_1 = \nabla f = (1,-2,3)$. This is at any point, as the gradient is constant.

   The normal vector to Q is $n_2 = (3,3,1)\times(2,-1,1) = (4,-1,-9)$. It is pretty clear that n_1 and n_2 are not parallel (they are not proportional), so neither are P and Q.

3. Consider the two functions $f : \mathbb{R} \to \mathbb{R}^3, f(t) = (\cos(t), \sin(t), t),\ g : \mathbb{R}^3 \to \mathbb{R}, g(x,y,z) = x^2 + y^2 + z^2$.

   (a) Use the chain rule to compute the derivative of $g \circ f$ at the point $t = \pi/4$.
   Solution: We compute that $f(\pi/4) = (\sqrt{2}/2, \sqrt{2}/2, \pi/4)$, and

   $$Df(\pi/4) = \begin{pmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \\ 1 \end{pmatrix}, Dg(\sqrt{2}/2, \sqrt{2}/2, \pi/4) = \begin{pmatrix} \sqrt{2} & \sqrt{2} & \pi/2 \end{pmatrix},$$

and finally that

$$D(g\circ f)(\pi/4) = Dg(\sqrt{2}/2, \sqrt{2}/2, \pi/4)Df(\pi/4) = \begin{pmatrix}\sqrt{2} & \sqrt{2} & \pi/2\end{pmatrix}\begin{pmatrix}-\sqrt{2}/2 \\ \sqrt{2}/2 \\ 1\end{pmatrix} = \pi/2.$$

Alternate Solution: The chain rule says that

$$\frac{d(g\circ f)}{dt} = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} + \frac{\partial g}{\partial z}\frac{dz}{dt}.$$

Then we compute the required partial derivatives and substitute in, recalling that $f(t) = (x(t), y(t), z(t))$.

(b) View the function f as describing a path in 3-space. Write an equation for the tangent line to this path at the point $(0, 1, \pi/2)$.

Solution: The line is those points described by

$$(0, 1, \pi/2) + t(-1, 0, 1), \qquad t \in \mathbb{R}.$$

4. An astronaut is floating in the middle of a nebula in outer space. The gas in this nebula is very hot, and she must decrease the temperature she experiences as quickly as possible. In a rectangular coordinate system centered on her, the temperature of the gas (in degrees Centigrade) is described by the equation

$$T(x, y, z) = -2x + \sin(x^2)y^2 + 2z + 78.$$

(a) Which direction should the astronaut go? (Note that the astronaut is located at $(0, 0, 0)$).

Solution: the astronaut should go in the direction of greatest decrease of the function, which is the negative of the gradient. So we compute $\nabla T(0, 0, 0) = (-2, 0, 2)$ *and suggest that the astronaut go in the direction of* $-\nabla T(0, 0, 0) = (2, 0, -2)$

(b) Would traveling in the direction of the vector $\mathbf{v} = (-1, -17, 1)$ increase or decrease the temperature she experiences?

Solution: We need to compute the directional derivative in the direction specified. It is

$$\nabla T(0, 0, 0) \cdot (-1, -17, 1)/||v|| = 4/||v|| > 0.$$

As this is positive, the temperature will increase in this direction.

5. Consider the graph of the function

$$f(x,y) = x^3(y^2-1) + (x-y)^3.$$

(a) Find the equation of the tangent plane to the graph at $P = (1,2)$.

Solution: Can do this by just recalling the linear approximation formula. Another way follows. The graph of f is the 0-level set of the function

$$g(x,y,z) = f(x,y) - z = x^3(y^2-1) + (x-y)^3 - z.$$

Thus the tangent plane can be recovered as the vectors orthogonal to the gradient of g. So we compute

$$\nabla g(1,2,2) = (12,1,-1)$$

and find the plane as those points $X = (x,y,z)$ which satisfy

$$0 = (X - (1,2,2)) \cdot (12,1,-1) = 12x + y - z - 12.$$

(b) Find a unit vector which is normal to the graph at P.

Solution: The above solution already finds a normal vector for us as $\nabla g(1,2,2) = (12,1,-1)$. We need merely normalize it to find a unit vector. The length of the gradient is $\sqrt{146}$, so the vector we want is $(12/\sqrt{146}, 1/\sqrt{146}, -1/\sqrt{146})$.

6. (a) Consider $h(x,y) = x^y$. Find the partial derivatives $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial y}$.

Solution: We compute that

$$\frac{\partial h}{\partial x} = yx^{y-1}, \quad \textit{and} \quad \frac{\partial h}{\partial x} = (\ln x)x^y.$$

(b) Use (a) and the Chain Rule to find

$$\frac{d}{dt}\left(f(t)^{g(t)}\right).$$

Solution: Let $z(t) = (f(t), g(t))$. Then $f(t)^{g(t)} = h \circ z(t)$. So by (a) and the chain rule, we get that

$$\begin{aligned} \frac{d}{dt}(h \circ z) &= \frac{\partial h}{\partial x}(f(t), g(t))\frac{df}{dt} + \frac{\partial h}{\partial y}(f(t), g(t))\frac{dg}{dt} \\ &= g(t)[f(t)]^{g(t)-1}f'(t) + \ln(f(t))[f(t)]^{g(t)}g'(t). \end{aligned}$$