Instructions: This was a 2 hour, closed notes, closed book, and pledged exam.

1. Find and classify all the critical points of

$$f(x, y) = \frac{1}{2}x^2 - xy + \frac{1}{3}y^3.$$

Solution: Since f is differentiable everywhere, the only critical points are the points where the gradient vanishes. $\nabla f(x, y) = (x - y, y^2 - x) = (0, 0)$ precisely at the points $(0, 0)$ and $(1, 1)$. Note $\frac{\partial^2 f}{\partial x^2} = 1$, $\frac{\partial^2 f}{\partial y \partial x} = -1$, and $\frac{\partial^2 f}{\partial y^2} = 2y$. Apply the second derivative test:

point	$\frac{\partial^2 f}{\partial x^2}$	D	*classification*
$(0, 0)$	1	-1	saddle
$(1, 1)$	1	1	strict local min

2. Let $g(x, y) = 2e^{-x} \cos y$.

(a) Find the quadratic Taylor polynomial for $g(x, y)$ around the point $(0, 0)$.

$$\begin{aligned}
g(0,0) &= 2 \\
\left.\frac{\partial g}{\partial x}\right|_{(0,0)} &= \left.-2e^{-x}\cos y\right|_{(0,0)} = -2 \\
\left.\frac{\partial^2 g}{\partial x^2}\right|_{(0,0)} &= \left.2e^{-x}\cos y\right|_{(0,0)} = 2 \\
\left.\frac{\partial^2 g}{\partial y \partial x}\right|_{(0,0)} &= \left.-2e^{-x}\sin y\right|_{(0,0)} = 0 \\
\left.\frac{\partial g}{\partial y}\right|_{(0,0)} &= \left.-2e^{-x}\sin y\right|_{(0,0)} = 0 \\
\left.\frac{\partial^2 g}{\partial y^2}\right|_{(0,0)} &= \left.-2e^{-x}\cos y\right|_{(0,0)} = -2
\end{aligned}$$

So the quadratic Taylor polynomial for $g(x, y)$ around $(0, 0)$ is

$$g(h_1, h_2) \approx 2 - 2h_1 + h_1^2 - h_2^2$$

(b) Use your answer in part (a) to estimate $2e^{-0.2} \cos 0.4$.
We just evaluate $g(0.2, 0.4) \approx 2 - 2(0.2) + (0.2)^2 - (0.4)^2 = 2 - .4 + .04 - .16 = 1.48$. (The actual value is approximately 1.508202).

3. A tank is in the shape of a half-cylinder of radius 2 and height 3. It is situated in $\mathbb{R}^3$, given by the inequalities $\sqrt{x^2 + y^2} \leq 2$, $y \geq 0$, and $0 \leq z \leq 3$. The temperature at the point (x, y, z) is given by

$$T(x, y, z) = 2yz^2\sqrt{x^2 + y^2} \quad {}^\circ\text{C}.$$

Find the average temperature in the tank.

Solution: We describe the tank in cylindrical coordinates as $0 \le r \le 2, 0 \le \theta \le \pi,$ *and* $0 \le z \le 3$. *Recall the formula* $[T]_{av} = \frac{\iiint_W T(x,y,z)\ dV}{\iiint_W\ dV}$.

We use cylindrical coordinates to compute

$$\begin{aligned}
\iiint_W T(x,y,z)\ dV &= \int_0^3 \int_0^\pi \int_0^2 2(r\sin\theta)(z^2)(r)\ r\ dr\ d\theta\ dz \\
&= 2\left(\int_0^3 z^2\ dz\right)\left(\int_0^\pi \sin\theta\ d\theta\right)\left(\int_0^2 r^3\ dr\right) \\
&= 2(2)(4)(9) = 144
\end{aligned}$$

The denominator (volume of region) is given by the formula Vol(W) $= \frac{\pi \cdot 2^2 \cdot 3}{2} = 6\pi$, *or by computing*

$$\iiint_W dV = \int_0^3 \int_0^\pi \int_0^2 r\ dr\ d\theta\ dz = \left(\int_0^3 dz\right)\left(\int_0^\pi d\theta\right)\left(\int_0^2 r\ dr\right) = (3)(\pi)(2) = 6\pi$$

Thus, $[T]_{av} = \frac{144}{6\pi} = \frac{24}{\pi}\ {}^\circ C$.

4. Let T be the triangle with vertices $(0,0), (1,1)$ and $(0,1)$ and let $f(x,y) = x\sin(y^3)$.

   (a) Find the correct limits of integration to **set up** $\iint_T f(x,y)\,dA$ as a double integral $\iint f(x,y)\,dx\,dy$.

   Solution: $\int_0^1 \int_0^y f(x,y)\ dx\ dy$.

   (b) Find the correct limits of integration to **set up** $\iint_T f(x,y)\,dA$ as a double integral $\iint f(x,y)\,dy\,dx$.

   Solution: $\int_0^1 \int_x^1 f(x,y)\ dy\ dx$.

   (c) Compute $\iint_T f(x,y)\,dA$.

   Solution: Use the set up from (a):

$$\begin{aligned}
\iint_T f(x,y)\ dA &= \int_0^1 \int_0^y x\sin(y^3)\ dx\ dy \\
&= \int_0^1 \frac{sin(y^3)}{2} x^2\Big|_{x=0}^{y}\ dy \\
&= \int_0^1 \frac{1}{2} y^2 \sin(y^3)\ dy \\
&= -\frac{\cos(y^3)}{6}\Big|_{y=0}^{1} \\
&= \frac{1-\cos 1}{6}
\end{aligned}$$

5. Find the maximum and minimum values obtained by $f(x,y) = x + y^2$ on the ellipse $x^2 + 3y^2 \leq 9$.

Solution: First, find critical points in the interior $x^2 + 3y^2 < 9$. *Note* $\nabla f(x,y) = (1, 2y)$ *is never* $(0,0)$, *so there are no critical points in the interior.*

Second, find critical points on the boundary $x^2 + 3y^2 = 9$ *using Lagrange Multipliers. Our constraint function is* $g(x,y) = x^2 + 3y^2$. *Solve* $\nabla f(x,y) = \lambda \nabla g(x,y)$, *i.e.* $(1, 2y) = \lambda(2x, 6y)$. *The second coordinate gives two possibilities:* $y = 0$ *or* $\lambda = 1/3$. *If* $y = 0$, *then* $x = \pm 3$ *(from the constraint* $x^2 + 3y^2 = 9$*). If* $\lambda = 1/3$, *then* $x = 3/2$ *(from* $1 = 2\lambda x$*), and the constraint gives* $y = \pm 3/2$. *There are four critical points to investigate:* $(\pm 3, 0)$ *and* $(3/2, \pm 3/2)$.

(x,y)	$f(x,y)$
$(3,0)$	3
$(-3,0)$	-3
$(3/2,3/2)$	$15/4$
$(3/2,-3/2)$	$15/4$

Thus the absolute maximum value of f *on boundary is* $15/4$, *and the absolute minimum value on the boundary is* -3.

Since there are no critical points from the interior, these maximum and minimum boundary values are also the maximum and minimum values throughout the entire region.

6. The region S is cut from a solid ball of radius 1 centered at the origin. S is the region cut by the inequalities $z \geq 0$ and $y \geq x$. (S is one-quarter of the entire ball, and contains the point $(0,1,0)$.)

The mass density of S at a point (x,y,z) is given by the function $\delta(x,y,z) = 30z^2$ kg/m^3.

(a) Find the total mass of S.

Solution: The total mass is given by $\iiint_S \delta(x,y,z)dV$. *Note that* S *is described in spherical coordinates by* $0 \leq \rho \leq 1, \pi/4 \leq \theta \leq 5\pi/4,$ *and* $0 \leq \phi \leq \pi/2$. *Thus*

$$
\begin{aligned}
\iiint_S \delta(x,y,z)\ dV &= \int_0^{\pi/2} \int_{\pi/4}^{5\pi/4} \int_0^1 30(\rho\cos\phi)^2\ \rho^2\ \sin\phi\ d\rho\ d\theta\ d\phi \\
&= 30 \left(\int_0^{\pi/2} \cos^2\phi \sin\phi\ d\phi \right) \left(\int_{\pi/4}^{5\pi/4} d\theta \right) \left(\int_0^1 \rho^4\ d\rho \right) \\
&= 30 \left(\frac{-\cos^3\phi}{3} \Big|_0^{\pi/2} \right) (\pi) \left(\frac{1}{5} \right) \\
&= 2\pi \text{ kg}
\end{aligned}
$$

(b) Find the average mass density of S.

Solution: Average mass density is

$$
[\delta]_{\text{av}} = \frac{\iiint_S \delta(x,y,z)\ dV}{\iiint_S\ dV} = \frac{2\pi \text{ kg}}{\text{Vol}(S)} = \frac{2\pi \text{ kg}}{\pi/3 \text{ m}^3} = 6 \text{ kg/m}^3
$$