Math 381: Final Exam

December 11, 2003

2pm - 5pm

Read the directions to each problem carefully. Show your work and answer in complete sentences when appropriate. This exam has 5 questions on 11 pages including this cover, blank pages for work and the table of Laplace transforms at the back.

Name:

Student ID:

On my honor, I have neither given nor received any aid on this examination.

Signed:____________________________________

Problem	Score
1	
2	
3	
4	
5	
Total	

Problem 1:(20 points)

a) What is the order of the following partial differential equation?

$$(x^3yz_{xx} + x^{58}z_{xy} + 12z)(\coth^{-1}(x^4)z_{xxx} - 5\cos(z_{xxyy})z_y) = 0$$

b) Write a specific example of a first order, linear partial differential equation in two variables.

c) Write a specific example of a first order, quasi-linear partial differential equation in two variables which is <u>not</u> linear.

d) Write Legendre's equation of order n.

e) Write Bessel's equation of order n.

Problem 1: (continued)

f) Define what it means for two functions f and g to be orthogonal over the interval $[0, 1]$ with respect to the weight function $\rho(x) = x$.

g) Suppose that f is an integrable function over the interval $[-\pi, \pi]$. How do we define the 'Fourier coefficients' of f?

h) Suppose that f is an integrable function over the interval $[-1, 1]$. How do we define the 'Legendre coefficients' of f?

i) Suppose that f is an integrable function over the interval $[0, 1]$. How do we define the 'Bessel coefficients' of f of order zero indexed on the roots of the equation $J_0(x) = 0$?

j) What does it mean for a function to be of 'exponential order'? Given a function g which is of exponential order, how does one define the Laplace transform of g?

Problem 2: (20 points) Choose two of the following free response questions to answer. Each will be worth 10 points and a concise paragraph or two should be sufficient. Be sure to indicate which questions you want graded.

Option One: State Parseval's theorem for Bessel series of order zero and explain its content. Also, give a linear algebra interpretation.

Option Two: Describe the geometry behind the 'method of characteristics' for solving first-order quasi-linear partial differential equations.

Option Three: Describe Gibbs' phenomenon and how it fits into the theory of convergence of Fourier series.

(This page to be used for question 2.)

Problem 3:

a) (12 points) Using the method of characteristics, find the general solution to the partial differential equation

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = z. \tag{1}$$

b) (5 points) Find the particular solution of equation (1) which satisfies the initial condition $z = x^3$ when $x = y$.

c) (3 points) How many solutions are there to equation 1 whose graphs contain the curve defined by $y = 1/x$, $z = 12$?

Problem 4:

a) (8 points) Set up a boundary value problem which describes the vibration of a circular drumhead which is *struck* at time $t = 0$. (Recall that in polar coordinates (r, θ) the Laplace operator is $\triangle z = \dfrac{\partial^2 z}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial z}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 z}{\partial \theta^2}$.)

b) (12 points) Use the method of separation of variables to solve this boundary value problem when the initial strike is described by the *rotationally symmetric* function

$$f(r, \theta) = -J_0(\lambda_1 \cdot r),$$

where J_0 is the Bessel function of the first kind and order zero, and λ_1 is the smallest positive number such that $J_0(\lambda_1) = 0$.

(This page to be used for question 4.)

Problem 5: (20 points) Solve the following boundary value problem using the Laplace transform method.

$$\begin{cases} \dfrac{\partial^2\phi}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2\phi}{\partial t^2} - \cos(\omega t), & 0 \le x < \infty, \quad 0 \le t < \infty, \\ \phi(0,t) = 0, & 0 \le t < \infty, \\ \phi(x,0) = 0, & 0 \le x < \infty, \\ \dfrac{\partial\phi(x,0)}{\partial t} = 0, & 0 \le x < \infty. \end{cases} \tag{2}$$

(This page is to be used for question 5.)