Solutions to the Final Examination
Math 381:
Introduction to Partial Differential Equations
Rice University, Fall 2003

Problem 1:

a) This is a fourth order equation.

b) Many correct responses are possible. An example would be

$$\frac{\partial z}{\partial x} + 12\frac{\partial z}{\partial y} = 3z.$$

c) Again, many correct answers are possible. An example would be

$$\frac{\partial z}{\partial x} + 12\frac{\partial z}{\partial y} = e^{3z}.$$

d) Legendre's equation of order n is

$$(1 - x^2) \cdot y''(x) - 2x \cdot y'(x) + n(n+1)y(x) = 0, \quad -1 \leq x \leq 1.$$

e) Bessel's equation of order n is

$$y''(x) + \frac{1}{x}y'(x) + \left(1 - \frac{n^2}{x^2}\right) y(x) = 0, \quad 0 \leq x \leq 1.$$

f) Two functions f and g are orthogonal over the interval $[0, 1]$ with respect to the weight function $\rho(x) = x$ when

$$\int_0^1 xf(x)g(x)\ dx = 0.$$

g) The Fourier coefficients of a function f are the numbers

$$A_0 = a_0/2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ dx,$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\ dx, \quad \text{and}$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\ dx.$$

h) If $P_n(x)$ denotes the nth Legendre polynomial, then the Legendre coefficients are the numbers

$$a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\ dx.$$

i) If $J_0(x)$ is the Bessel function of the first kind and order zero and its positive roots are enumerated as $\lambda_1 < \lambda_2 < \lambda_3 < \dots$, then the coefficients asked for are the numbers

$$a_n = \frac{2}{[J_0'(\lambda_n)]^2} \int_0^1 x f(x) J_0(\lambda_n \cdot x)\ dx.$$

j) A function is said to be of exponential order if there are positive constants M, s_0, T such that

$$|f(t)| < M e^{s_0 t}, \quad \text{for all } t > T.$$

If g is of exponential order, we define the Laplace transform of g by

$$\overline{g}(s) = \int_0^\infty e^{-st} f(t)\ dt,$$

for all s large enough that this integral exists.

Problem 2: We give outlines of answers for all three options.

Option One: The statement of Parseval's Theorem for Bessel series of order zero was the content of a homework exercise in problem set # 6. See the relevant exercise for a statement. The content of this result is that the mean square error of a partial sum of Bessel series approximation approaches zero as the number of terms increases, so that, at least in a mean square error sense, Bessel series do a pretty good job of approximating an arbitrary function. In terms of linear algebra, it says that we can view the scaled Bessel functions $J_0(\lambda_i \cdot x)$ form a complete basis of the infinite dimensional vector space of functions on the interval $[0, 1]$.

Option Two: The method of characteristics is based on the following geometric principle. Given a quasi-linear first order partial differential equation

$$P(x, y, z)\frac{\partial z}{\partial x} + Q(x, y, z)\frac{\partial z}{\partial y} = R(x, y, z),$$

we reinterpret the equation as a statement that the normal vector $(u_x, u_z, -1)$ to an integral surface $u(x, y, z) = C$ is perpendicular to the *characteristic vector field* $V = (P(x, y, z), Q(x, y, z), R(x, y, z))$. This means that the integral curves of this vector field must lie entirely inside any integral surface which they intersect. So, to solve the partial differential equation, we need only find out what these integral curves are and learn to bundle them together into nice families to make surfaces.

The integral curves of the characteristic vector field are called *characteristic curves* and are defined by a system of ordinary differential equations. Solving one of these ordinary differential equations determines a family of surfaces in space of the form $u_i(x, y, z) = C_i$. A characteristic curve is formed when we take the intersection of a surface from one family with a surface of the other family. As a practical matter, one needs to decide on the functional relationship between the parameter C_1 which tells you which surface from the first family to use, and the parameter C_2 which tells you which surface from the second family to use. This gives rise to an "arbitrary" function in the general solution to the partial differential equation.

Option Three: The Gibbs phenomenon is a quantification of the observation that a Fourier series has trouble approximating a function uniformly well near a point of discontinuity. Gibbs investigated the function $\Psi(x)$ defined as an odd, 2π-periodic extension of $\frac{1}{2}(\pi - x), 0 < x < \pi$. Gibbs showed that near the discontinuity zero, partial sums of the Fourier series for Ψ always miss the function and form waves. More importantly, as the number of terms increases, the size of the "misses" does not go to zero! Gibbs calculated the limiting value of the size of the misses precisely, the first one being about $(0.09)\pi$, or about 9% of the size of the jump in Ψ at $x = 0$.

Gibbs' phenomenon is important because it shows the limits of how well one can approximate a discontinuous function using Fourier series. First, it shows that we never get uniform convergence of the Fourier series of a function with a jump. Most importantly, Gibbs' phenomenon gives a measure of how bad the error will always be, and thus gives us some way to deal with convergence issues near discontinuities. Even if things must be bad, we at least know precisely how bad they will be.

Problem 3:

a) This equation was studied in homework assignment # 1. The general solution is of the form $z = x \cdot f(xy)$, where f is an arbitrary C^1 function.

b) We substitute in the required condition to find $x^3 = z = x \cdot f(xy) = x \cdot f(x^2)$. This means that we must take $f(u) = u$, so our solution is $z = x^2 y$.

c) The curve $y = 1/x$ is the projection of a characteristic curve. (The other part of the condition does not matter here, the equation is linear, not just quasi-linear.) This means there is the standard duality. If our curve is a characteristic, there are infinitely many solutions, and if not, there are none. Comparing with the work for part (a), we see that the curve in question is not a characteristic, so there is no solution meeting these conditions.

Problem 4:

a) We have several conditions to meet.

$$\begin{cases} \dfrac{\partial^2 z}{\partial t^2} = a^2 \left(\dfrac{\partial^2 z}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial z}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 z}{\partial \theta^2} \right), & 0 \le t < \infty,\ 0 \le r \le 1,\ 0 \le \theta \le 2\pi, \\ z(t, r, \theta + 2\pi) = z(t, r, \theta), & 0 \le t < \infty,\ 0 \le r \le 1, \\ z(t, 1, \theta) = 0, & 0 \le t < \infty,\ 0 \le \theta \le 2\pi, \\ z(0, r, \theta) = 0 & 0 \le \theta \le 2\pi,\ 0 \le r \le 1, \\ \dfrac{\partial z}{\partial t}(0, r, \theta) = \phi(r, \theta) & 0 \le \theta \le 2\pi,\ 0 \le r \le 1. \end{cases}$$

b) Since the boundary values are rotationally symmetric, we seek a solution which is rotationally symmetric, too. This drops the final term of the first equation from consideration, and removes the periodicity condition.

We try to separate the variables by setting $z(t, r) = T(t) \cdot R(r)$. The differential equation then forces that

$$RT'' = a^2 \left(R''T + \frac{1}{r}R'T \right).$$

This is equivalent to

$$\frac{T''}{a^2 T} = \frac{R'' + R'/r}{R}.$$

Since the right hand side is independent of t and the left hand side is independent of r, this quantity must be a constant. Calling the constant C, we get the following pair of ordinary differential equations to solve.

$$\begin{cases} T'' = Ca^2 T, & 0 \le t < \infty \\ R'' + \dfrac{1}{r}R' - CR = 0, & 0 \le r \le 1. \end{cases}$$

The first equation has either trigonometric functions or exponentials as general solution, depending on the sign of C. Since the problem is about a physical drumhead, it is more appropriate to have the bounded trigonometric functions than the unbounded exponential functions. So we take $C = -\lambda^2$ to be negative.

Thus the first equation has general solution of the form $A\cos(a\lambda t) + B\sin(a\lambda t)$. The second equation has only one solution which is bounded for all r, it is $J_0(\lambda r)$, where J_0 denotes the Bessel function of the first kind and order zero. We deduce that our fundamental product solution should have the form

$$J_0(\lambda r)\left(A\cos(a\lambda t) + B\sin(a\lambda t)\right).$$

Now we apply the remaining boundary conditions. The condition that the drumhead is initially still implies that $A = 0$. So our fundamental solution has the form $J_0(\lambda r)\sin(a\lambda t)$. Then, the condition about the edge of the drumhead being fixed means that λ must be a root of the equation $J_0(x) = 0$.

So if we enumerate the roots of $J_0(x) = 0$ as $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$, we see that the formal solution of our equations so far is the infinite sum

$$z(t, r) = \sum_{i=1}^{\infty} A_i J_0(\lambda_i r) \sin(a\lambda_i t).$$

In order to meet the last boundary condition, we must choose the coefficients A_i so that

$$-J_0(\lambda_1 \cdot r) = f(r, \theta) = z_t(0, r) = \sum_{i=1}^{\infty} A_i a\lambda_i J_0(\lambda_i r),$$

that is, we should have that $a\lambda_i A_i$ is the ith Bessel series coeffiecient of $-J_0(\lambda_1 \cdot r)$. But this expression is its own Bessel series, so we don't need to compute it. (If you try to compute it, the orthogonality of Bessel series immediately gives that all the coefficients vanish except for the first one.) Therefore, we must choose $A_1 = -1/(a\lambda_1)$, and all other A_i are zero. The solution to our problem is the function

$$z(t, r) = \frac{-1}{a\lambda_1} J_0(\lambda_1 r) \sin(a\lambda_1 t).$$

Problem 5: Taking the Laplace transform of the differential equation we find

$$\frac{\partial^2 \overline{\phi}}{\partial x^2} = \frac{1}{c^2}\left(-\frac{\partial \phi}{\partial t}(x, 0) - s\phi(x, 0) + s^2\overline{\phi}(x, s)\right) - \frac{s}{s^2 + \omega^2} = \frac{s^2}{c^2}\overline{\phi}(x, s) - \frac{s}{s^2 + \omega^2}.$$

This is an inhomogeneous ordinary differential equation in x, and the inhomogeneous term is a constant (in x). The associated homogeneous equation has solution $A(s)e^{sx/c} + B(s)e^{-sx/c}$, so we look for a solution to our equation of the form

$$A(s)e^{sx/c} + B(s)e^{-sx/c} + C(s).$$

A simple check shows that this works if we choose $C(s) = \frac{c^2}{s(s^2 + \omega^2)}$. It is impossible for the first term to be a transform of any function since it does not decay to zero as s increases for large values of x. Therefore, our solution has transform of the form

$$\overline{\phi}(x, s) = B(s)e^{-sx/c} + \frac{c^2}{s(s^2 + \omega^2)}.$$

To find $B(s)$, we check the remaining boundary condition. Its transform is $\overline{\phi}(x, s) = 0$, which means that $B(s) = -\frac{c^2}{s(s^2 + \omega^2)}$. This means that the Laplace transform of our solution is

$$\overline{\phi}(x, s) = -\frac{c^2}{s(s^2 + \omega^2)}e^{-sx/c} + \frac{c^2}{s(s^2 + \omega^2)}.$$

We must find the inverse Laplace transform of this expression to finish the problem. The inverse transform of $\dfrac{c^2}{s(s^2+\omega^2)}$ is

$$\frac{c^2}{\omega}\int_0^t \sin(\omega u)\ du = -\frac{c^2}{\omega^2}\cos(\omega t) + \frac{c^2}{\omega^2}.$$

So by the translation property of Laplace transforms, we see that the answer is

$$\begin{aligned}\phi(x,t) &= -\left\{\begin{array}{cc} 0, & 0 \le t \le x/c \\ -\dfrac{c^2}{\omega^2}\cos(\omega(t-x/c)) + \dfrac{c^2}{\omega^2}, & t > x/c \end{array}\right\} - \frac{c^2}{\omega^2}\cos(\omega t) + \frac{c^2}{\omega^2} \\ &= \left\{\begin{array}{cc} \dfrac{c^2}{\omega^2} - \dfrac{c^2}{\omega^2}\cos(\omega t), & 0 \le t \le x/c \\ \dfrac{c^2}{\omega^2}\cos(\omega(t-x/c)) - \dfrac{c^2}{\omega^2}\cos(\omega t), & t > x/c. \end{array}\right.\end{aligned}$$

It is not difficult to check that this satisfies all of the conditions. Note that this solution has a "corner" in it corresponding to when the front of the travelling wave passes by.